Constrained Iterative Restoration Algorithms

RONALD W. SCHAFER, FELLOW, IEEE, RUSSELL M. MERSEREAU, SENIOR MEMBER, IEEE, AND MARK A. RICHARDS, STUDENT MEMBER, IEEE

Invited Paper

Abstract—**This paper describes a rather broad class of iterative signal restoration techniques which can be applied to remove the effects of many different types of distortions. These techniques also allow for the incorporation of prior knowledge of the signal in terms of the specification of a constraint operator. Conditions for convergence of the iteration under various combinations of distortions and constraints are explored. Particular attention is given to the use of iterative restoration techniques for constrained deconvolution, when the distortion bandlimits the signal and spectral extrapolation must be performed. It is shown that by predistorting the signal (and later removing this predistortion) it is possible to achieve spectral extrapolation, to broaden the class of signals for which these algorithms achieve convergence, and to improve their performance in the presence of broad-band noise.**

I. Introduction

THE RECOVERY or restoration of a signal that has been distorted is one of the most important problems in signal processing. Some examples are the recovery of the input to a linear shift-invariant system from its output (deconvolution), the restoration of a multidimensional signal from its projections, the recovery of the input to a nonlinear or shift-varying system from its output, and the extrapolation of a signal from a finite segment of that signal. In all these cases an appropriate mathematical representation is

$$y = Dx \tag{1}$$

where x is the unknown input signal, y is the known output signal, and D is a known distortion operator or transformation. In (1), x and y might represent continuous signals of one or more dimensions, in which case x and y would be functions of one or more continuous variables, or they might be discrete signals, in which case x and y would be sequences or arrays of numbers. D should be thought of as a general operator which will map either functions into functions or sequences into sequences. The problem of signal restoration is simply that of recovering x given y and D. From this point of view, equation (1) is a functional equation that must be solved for x.

One approach to solving for x is to find the inverse operator D^{-1} such that

$$x = D^{-1}y. \tag{2}$$

In many cases of practical interest, however, it may be difficult, or indeed impossible, to determine and implement the inverse operator, or we may have available only an approximation to the operator D. Implementation of the inverse operator based on incomplete knowledge of D may be quite unsatisfactory. Even if D^{-1} can be approximated and implemented, the result of applying it to the signal y may differ radically from the true solution if y is known imprecisely due to such uncertainties as additive noise. In addition, the distortion operator D may be such that many inputs will produce the same output signal y in which case the inverse operator does not exist. This situation occurs if D is the bandlimiting operator. In such cases, limited prior knowledge of the properties of x may be useful in removing the ambiguity. By definition, however, D^{-1} is determined solely by D, and the properties of x are not considered. Thus there are many disadvantages in the direct implementation of inverse operators for signal restoration.

For these reasons, alternative approaches to signal restoration are of interest. One approach that is particularly attractive for computer implementations is the method of successive approximations. This approach, which has a long mathematical history, is typically based upon an iteration equation of the form

$$x_{k+1} = Fx_k \tag{3}$$

where F is an operator which is obtained from (1). It is not necessary, however, that F depend only upon D. As we shall see, it may also incorporate constraints based upon known properties of the desired solution. In general the iteration equation is not unique; many different iteration equations can be derived for a given distortion operator and set of signal constraints.

Such iterations are useful if it can be shown that the sequence of approximations $\{x_k\}$ converges to a unique solution. Although convergence generally requires an infinite number of iterations, acceptable approximations can usually be found for finite values of k. Practical interest in iterations such as these stems from the great flexibility which is available for mixing constraints and distortions. Often the disadvantage of slow convergence can be offset by the fact that it is not necessary to determine the inverse of the distortion operator. This is particularly significant for nonlinear or shift-varying (e.g., time-varying) distortion operators.

These advantages have led many researchers, including ourselves, to investigate a variety of iterative restoration procedures. One of the purposes of this paper is to show that these different schemes are all special cases of a general iteration equation that incorporates constraints on the desired solution in a very straightforward manner.

The paper begins with the derivation of the general iteration

Manuscript received July 7, 1980; revised December 23, 1980. This work was supported by the Joint Services Electronics Program under Contract DAAG29-76-G-0226, and by the John and Mary Franklin Foundation.

The authors are with the School of Electrical Engineering, Georgia Institute of Technology, Atlanta, GA 30332.

0018-9219/81/0400-0432$00.75

formula and a discussion of its convergence properties. This theoretical framework is developed in Section II. In Section III, several iterative procedures are shown to be special cases of the general iteration formula. The framework of Section II can also be used as a basis for generating new iterative procedures for handling other combinations of distortions and constraints. Section IV of the paper examines a particular subclass of constrained iterative deconvolution procedures. In particular it is shown that for signals that are nonnegative or that have finite support (e.g., are time-limited or space-limited), the incorporation of appropriate constraints leads to a sensible extrapolation of the Fourier transform so as to restore high frequency information lost when the distortion is a bandlimiting operator. Examples of the application of iterative procedures to both synthetic signals and gamma-ray spectrum data illustrate the convergence properties and the effects of additive noise. The final section of the paper summarizes the important results and offers suggestions of new applications of the basic iterative procedures.

II. A Basic Iteration Equation

A. Derivation

The restoration of x from y by an iterative procedure is facilitated by the use of prior knowledge of the properties of the desired solution. For example, we may know that x is a bandlimited signal, or that it is time-limited or space-limited, or we may know on physical grounds that x can have only nonnegative values. A convenient way of expressing such prior knowledge is to define a constraint operator C, such that

$$x = Cx \tag{4}$$

if and only if x satisfies the constraint. For example, if x is a discrete one-dimensional signal, known to be nonnegative, then C could be defined by the positivity operator P; i.e.,

$$C[x(n)] \triangleq P[x(n)] = \begin{cases} x(n), & \text{if } x(n) \geqslant 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Similarly, if x is an analog signal, known to be bandlimited to frequencies below Ω_c, then the appropriate constraint operator can be defined by

$$C[x(t)] \triangleq B[x(t)] = \int_{-\infty}^{\infty} x(\tau) \frac{\sin[\Omega_c(t-\tau)]}{\pi(t-\tau)} \, d\tau. \tag{6}$$

In these cases, signals with the prescribed properties are not changed by the constraint operators, and signals not having the prescribed properties are converted into signals which do have those properties.

Using such a representation for *a priori* signal constraints, equation (1) can be expressed as

$$y = DCx \tag{7}$$

where the operator DC is the concatenation of C followed by D.

One approach toward obtaining an iteration equation follows from combining (4) and (7) to obtain the identity

$$x = Cx + \lambda(y - DCx) \tag{8}$$

where λ is either a constant parameter, a function of the independent variables, or a function of x. Equation (8) is clearly in the form

$$x = Fx \tag{9}$$

where the operator F is defined by

$$Fx = \lambda y + Cx - \lambda DCx = \lambda y + Gx \tag{10}$$

with

$$G = (I - \lambda D)\, C \tag{11}$$

where I is the identity operator. The signal x that satisfies (9) is called a fixed point of the transformation F [1]–[3]. A standard technique for finding such solutions is the method of successive approximations based upon the iteration equation

$$x_{k+1} = Fx_k = \lambda y + Gx_k. \tag{12}$$

We shall later see that λ can sometimes be used to control the rate of convergence of the iteration. In many cases it is convenient, even advantageous, to choose the initial approximation as $x_0 = \lambda y$. However, this is not necessary and x_0 can, in general, be any signal in the space of functions containing the desired solution x.

The convergence properties and the uniqueness of the final result of such an iterative procedure are, of course, a major consideration in practical applications. As we shall see, the theory of functional analysis provides powerful theorems that can be directly applied to the solution of (9) using the iteration of (12).

B. Convergence

We shall begin by assuming that the signals x and the outputs of the iteration x_k are members of a complete normed linear space with norm $\|x\|$ defined, for example, by

$$\|x\| = \left\{ \int_{-\infty}^{\infty} |x(t)|^2 \, dt \right\}^{1/2} \tag{13}$$

for continuous one-dimensional signals, and by

$$\|x\| = \left\{ \sum_{n=-\infty}^{\infty} |x(n)|^2 \right\}^{1/2} \tag{14}$$

for discrete one-dimensional signals. Suppose that

$$\|Fx_i - Fx_j\| \leqslant r\|x_i - x_j\| \tag{15}$$

for x_i and x_j in some closed subspace of the space of signals. If $0 \leqslant r < 1$, the operator F is said to be a *contraction mapping* (or simply a *contraction*) in that subspace [1]–[3]. If $r = 1$ the operator is said to be *nonexpansive*, and if $r = 1$ and (15) holds with equality only if $x_i = x_j$, then the operator F is *strictly nonexpansive* [3].

Since the norm can be interpreted as the distance between two signals, we can say that contraction operators (systems) have the property that the distance between two signals tends to decrease as the signals are transformed by the operator. For linear systems, this is roughly equivalent to saying that the gain of a contractive operator is less than unity.

If the operator F is a contraction in some subspace, then it has a unique fixed point x in that subspace such that $x = Fx$. Furthermore, every sequence of successive approximations defined by (12) converges to x for every choice of the starting signal x_0 in the subspace; i.e., $x_k \to x$ as $k \to \infty$. The above two sentences state a form of the well known *contraction*

mapping theorem of functional analysis [2], [3]. A further consequence of (15) is

$$\|x - x_k\| \leqslant \frac{r^{k+1}}{1-r} \|x - x_0\| \tag{16}$$

for any x_0 in the subspace [2], [3]. That is, every sequence of iterations converges geometrically to the unique fixed point x in the sense that $\lim_{k\to\infty} \|x - x_k\| = 0$.

This is a very powerful theorem which not only guarantees convergence of the iteration in (12), but also guarantees the existence and uniqueness of the solution.

If the operator is only nonexpansive, the situation is not as nice. Nonexpansive operators may have many fixed points, and the method of successive substitutions may not converge. If, however, the operator is strictly nonexpansive, there may be a unique fixed point and iterations based upon the method of successive substitutions may converge [3].

For the iteration scheme in question, F is defined by (10) and (11). Thus the iteration of (12) converges if F is a contraction. From (10) it follows that

$$\|Fx_i - Fx_j\| = \|Gx_i - Gx_j\|. \tag{17}$$

Thus the iteration converges if G is a contraction. The operator G, given by (11), is seen to involve both the distortion operator D and the constraint operator C as well as the quantity λ, which we are free to choose to guarantee that G is a contraction. It may also be possible to choose λ so that the rate of convergence is optimized. The details of the proof that G is a contraction depend upon the specific properties of D and C. Numerous examples are discussed in Section III.

C. Noniterative Solution

When G is a linear operator, it is possible to obtain a closed-form expression of the form $x_k = H_k x_0$, which gives the result of k iterations of (12) directly in a single operation. To see this, we substitute $x_0 = \lambda y$ into (12) to obtain

$$x_{k+1} = x_0 + Gx_k. \tag{18}$$

When G is a linear operator, it is easy to see that repeated application of (18) leads to

$$x_k = x_0 + Gx_0 + G^2 x_0 + \cdots + G^k x_0 \tag{19}$$

where G^i means that the operator G is applied i times. Since G is linear, we can express x_k as

$$x_k = H_k x_0 \tag{20}$$

where H_k is the operator

$$H_k = \sum_{i=0}^{k} G^i \tag{21}$$

and G^0 is understood to be the identity operator I. If $\|Gx\| \leqslant r\|x\|$ where $0 \leqslant r < 1$, which is true if G is a contraction, then it can be shown [2] that

$$H_k = (I - G)^{-1}(I - G^{k+1}). \tag{22}$$

Equation (22) gives an expression for the direct transformation from the initial to the kth approximation. Furthermore, if $\|Gx\| \leqslant r\|x\|$, it is easily shown that $\|G^{k+1}x\| \leqslant r^{k+1}\|x\|$, so that if $0 \leqslant r < 1$, then $\|G^{k+1}x\| \to 0$ as $k \to \infty$. This suggests that the solution x could be found directly ($x = Hx_0$) using the operator

$$H = (I - G)^{-1} \tag{23}$$

It is worth pointing out that (23) follows directly from (18) by substituting $x = x_{k+1} = x_k$ and solving for x in terms of x_0. This can be done without the assumption of linearity required to obtain (22).

Obviously, the use of (22) and (23) can avoid a great deal of computation; however, the utility of these equations depends upon the ease with which the operator $(I - G)^{-1}$ can be implemented. It may be no easier to implement $(I - G)^{-1}$ than to implement D^{-1} directly.

D. Error Analysis

There are three basic sources of error in using the iteration of (12). They are: 1) truncation of the sequence of iterations; 2) inaccuracies in measurements of y or in the assumed distortion and constraint operators; and 3) imperfections in the implementation of the iteration. We shall discuss 1) and 2) in this section and 3) will be discussed in Section IV.

The effect of truncation of the sequence of iterations can be assessed by considering (16), which states that the norm of the error decreases at least exponentially. In cases where r can be accurately estimated, equation (16) could serve as a quantitative bound on the error. However, equation (16) may not be very useful since it may be difficult to estimate r, or the estimate may be signal dependent. Also, in some cases, convergence may be faster than implied by the bound in (16). Some quantitative results for a class of deconvolution algorithms operating on impulsive signals are given in Section IV.

Errors also result when the available output does not satisfy (7). Thus, if y' is the observed output due to an input x, and y is the output of the assumed model due to that same input, then using the operator $G = (I - \lambda D)C$ with these outputs, the iterative procedure will converge respectively to x' and x satisfying the equations

$$x' = \lambda y' + Gx' \tag{24}$$

$$x = \lambda y + Gx. \tag{25}$$

Taking the difference of these two equations, we see that

$$\|x' - x\| = \|\lambda(y' - y) + Gx' - Gx\|. \tag{26}$$

Using the triangle inequality for the norm and the assumption that G is a contraction leads to the inequality

$$\|x' - x\| \leqslant \lambda \frac{\|y' - y\|}{1 - r} \tag{27}$$

where $\|Gx' - Gx\| \leqslant r\|x' - x\|$. This gives a bound on the error due to an inadequate representation of the output.

A simple example occurs when

$$y' = y + e \tag{28}$$

i.e., when there is additive noise in addition to the distortion modeled by the operator D. If (28) is substituted into (27), we obtain

$$\|x' - x\| \leqslant \frac{\lambda\|e\|}{1 - r} \tag{29}$$

which gives a bound on the error of the iteration in terms of the norm of the noise.

Another way in which errors can result is if the actual distortion operator differs from the assumed distortion operator; i.e.,

$$y' = D'x \tag{30}$$

Substituting (30) into (27) gives

$$\|x' - x\| \leqslant \frac{\lambda \|D'x - Dx\|}{1 - r}. \tag{31}$$

An interesting and useful result can be obtained for the case where D and D' are linear shift invariant operators so that $Dx = h * x$ where $*$ denotes convolution. Using Parseval's theorem we note for one-dimensional finite energy analog signals that

$$\|D'x - Dx\| = \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, |H'(\omega) - H(\omega)|^2 \, d\omega\right\}^{1/2} \tag{32}$$

where $X(\omega)$, $H'(\omega)$, and $H(\omega)$ are the Fourier transforms of x, h', and h respectively. By replacing $X(\omega)$ in (32) by its maximum value and substituting the result into (31) we obtain

$$\|x' - x\| \leqslant \frac{\lambda \|h' - h\|}{1 - r} \max_{\omega} |X(\omega)|. \tag{33}$$

Alternatively, we can replace $|H'(\omega) - H(\omega)|$ in (32) by its maximum value to obtain

$$\|x' - x\| \leqslant \frac{\lambda \|x\|}{1 - r} \max_{\omega} |H'(\omega) - H(\omega)|. \tag{34}$$

In the important special case of bandlimited signals, the above bounds on $\|x' - x\|$ can be converted to bounds on $|x' - x|$ using the inequality,

$$|x| < \sqrt{\Omega_c/\pi} \, \|x\| \tag{35}$$

which holds for any signal whose Fourier transform vanishes above a frequency Ω_c [4], [5]. It should be clear that although (32)–(35) were obtained for analog one-dimensional signals, very similar equations could be obtained for discrete or multidimensional signals.

These bounds indicate that the iterative procedure is stable in the sense that if the errors in y' are small, then the error in x' will also be small. Unfortunately, such bounds are not particularly useful in predicting the detailed properties of the resulting error. An example illustrating this point is given in Section IV. However, these inequalities do suggest that the key to reducing the error $\|x' - x\|$ is to reduce $\|y' - y\|$, i.e., we should try to make the given signal and assumed distortion model be consistent. In Section IV it is shown that this can sometimes be achieved by increasing the distortion in a known way prior to the restoration iteration.

III. A Survey of Iteration Schemes

Many researchers have investigated iterative restoration schemes that can be shown to be special cases of (12). In this section we shall call attention to a number of these schemes, and we shall show how the results of the previous section can be applied to determine conditions for convergence. Although we have uncovered many references to iterative approaches that fit into the general framework of Section II, we are not willing to speculate as to who first applied the iterative approach to specific signal restoration problems. Therefore, instead of a chronological order, the order of presentation in this section is such that schemes that have much in common, such as similar constraint or distortion operators, are discussed together.

A. Bandlimited Nonlinear Distortion; Bandlimited Constraint

Landau and Miranker [4], [5] considered the problem of recovering a bandlimited signal that has been distorted by a bandlimited nonlinear system. This problem was also considered by Zames [6]. Sandberg [7] has considered a more general version of this problem. In this case, the constraint operator was the bandlimiting operator B as defined in (4), and the distortion operator was assumed to be a memoryless nonlinearity with output Φx, followed by bandlimiting; i.e.,

$$y = B\Phi x. \tag{36}$$

This distortion operator was proposed as a model for transmission of companded signals over a telephone channel. It may also be an appropriate model for nonlinearities inherent in many image formation and digitization systems [8]. After nonlinear companding, the signal, Φx, is no longer bandlimited to the original frequency band. In fact, it may not be bandlimited at all. If this signal is later bandlimited, the question naturally arises as to whether it is possible to recover x from y. Landau and Miranker [4], [5] showed that x could indeed be recovered from y using the following iteration

$$x_0 = \lambda y \tag{37a}$$

$$x_{k+1} = \lambda y + x_k - \lambda B\Phi x_k. \tag{37b}$$

By substituting $D = B\Phi$ and $C = B$ into (12) and expanding the operator $G = (I - \lambda D)C$, we obtain

$$x_0 = \lambda y \tag{38a}$$

$$x_{k+1} = \lambda y + Bx_k - \lambda B\Phi Bx_k. \tag{38b}$$

Note, however, that any signal passing through the distortion operator D is bandlimited. Therefore, since $x_0 = \lambda B\Phi x$ is also bandlimited, it follows that all of the approximations, x_k are automatically bandlimited and that Bx_k can be replaced by x_k in (38b), thereby obtaining (37b).

Landau and Miranker [4], [5] studied the conditions for convergence and uniqueness in a style that we have followed in Section II for the general formulation. They showed that for convergence, i.e., for $G = (I - \lambda B\Phi)B$ to be a contraction, it must be true that $\Phi'(x) = d\Phi/dx$ satisfy the inequality

$$\max_{x} |1 - \lambda\Phi'(x)| < 1 \tag{39}$$

from which it follows that Φ' and λ must satisfy the conditions

$$0 < \Phi'(x) < M < \infty \tag{40a}$$

$$\lambda < 2/M. \tag{40b}$$

Error bounds similar to those of Section II-D are also given in [4] and will not be repeated here. The results of an analog implementation were also reported. Although we have not encountered any subsequent applications of this iteration procedure to discrete signals, it should be pointed out that the

inherent bandlimited constraint makes this scheme readily applicable to sampled signals.

B. Time Limiting Distortion; Bandlimited Constraint

Several authors, including Papoulis [9], Sabri and Steenart [10], and Wiley [11] have considered the problem of iteratively extrapolating from a finite duration segment of a bandlimited signal. In this case the distortion operator for discrete signals (sequences) is the linear time-limiting operator

$$y(n) = T[x(n)] = \begin{cases} x(n), & -\infty < n_a \leqslant n \leqslant n_b < \infty \\ 0, & \text{otherwise.} \end{cases} \tag{41}$$

At this point it is necessary to call attention to an obvious but important point concerning bandlimited signals. First, it is well known that if an analog signal is bandlimited, it can be represented without error by samples taken at a rate which is at least twice the highest frequency in the signal. However, if we sample at higher than the Nyquist rate, it would be appropriate to say that the sequence is bandlimited since its Fourier transform will be zero for frequencies in some range $\Omega_c < \omega \leqslant \pi/T$. Indeed, if we wish to apply a bandlimitation constraint to a discrete signal representation (as in Section III-A), then the signal must be sampled at a rate which is higher than the Nyquist rate. In this case, the discrete bandlimiting constraint operator is defined by the relation

$$B[x(n)] = \sum_{m=-\infty}^{\infty} x(m) \frac{\sin[\Omega_c T(n-m)]}{\pi(n-m)} \tag{42}$$

where T is the sampling period and $\Omega_c T < \pi$.

The iteration equation for these conditions is

$$x_0 = y \tag{43a}$$

$$x_{k+1} = y + (I - T)Bx_k. \tag{43b}$$

In this case the parameter λ which appeared in (12) is set to unity since $y(n) = x(n)$ for $n_a \leqslant n \leqslant n_b$ and it makes no sense to alter these correct samples. In this case, the operator $G = (I - T)B$ is a nonexpansive operator as we shall show below.

Since $(I - T)$ is a linear operator, we can write for the Euclidean norm

$$\|Gx_i - Gx_j\| = \|(I - T)(Bx_i - Bx_j)\| = \left\{ \sum_{n=-\infty}^{\infty} (1 - w(n))(\tilde{x}_i(n) - \tilde{x}_j(n))^2 \right\}^{1/2} \tag{44}$$

where

$$w(n) = \begin{cases} 1, & n_a \leqslant n \leqslant n_b \\ 0, & \text{otherwise} \end{cases} \tag{45}$$

and $\tilde{x}_i = Bx_i$ and $\tilde{x}_j = Bx_j$. This sum can be decomposed to give

$$\|Gx_i - Gx_j\| = \left\{ \sum_{n=-\infty}^{\infty} (\tilde{x}_i(n) - \tilde{x}_j(n))^2 - \sum_{n=n_a}^{n_b} (\tilde{x}_i(n) - \tilde{x}_j(n))^2 \right\}^{1/2} \tag{46}$$

Clearly we can write

$$\|Gx_i - Gx_j\| \leqslant r_1 \|Bx_i - Bx_j\| \tag{47}$$

where $0 \leqslant r_1 \leqslant 1$ with strict equality holding in (47) only if $\tilde{x}_i(n)$ and $\tilde{x}_j(n)$ are identical in the interval $n_a \leqslant n \leqslant n_b$. Now using Parseval's theorem, we obtain

$$\|Bx_i - Bx_j\| = \left\{ \frac{T}{2\pi} \int_{-\Omega_c}^{\Omega_c} |X_i(e^{j\omega T}) - X_j(e^{j\omega T})|^2 \, d\omega \right\}^{1/2}. \tag{48}$$

where $X_i(e^{j\omega T})$ and $X_j(e^{j\omega T})$ are the Fourier transforms of the sequences $x_i(n)$ and $x_j(n)$. From (48) it follows that

$$\|Bx_i - Bx_j\| \leqslant r_2 \|x_i - x_j\| \tag{49}$$

where $0 \leqslant r_2 \leqslant 1$, and $r_2 = 1$ only if $X_i(e^{j\omega T})$ and $X_j(e^{j\omega T})$ are identical in the band $\Omega_c < |\omega| \leqslant \pi/T$. Thus the discrete operators $(I - T)$ and B are both nonexpansive and

$$\|Gx_i - Gx_j\| \leqslant r_1 r_2 \cdot \|x_i - x_j\| \tag{50}$$

where $0 \leqslant r_1 r_2 \leqslant 1$. That is, the operator G is nonexpansive.

The corresponding manipulations of the norms for the continuous or analog case would be essentially the same as the above except the sums in (44) and (46) would be replaced by integrals. In this case it can be shown that r_1 and r_2 cannot be simultaneously equal to unity and thus the operator $(I - T)B$ is strictly nonexpansive and convergence to a unique fixed point is guaranteed [3]. To see this, note that strict equality holds only if Bx_i and Bx_j are identical over a finite interval. This is impossible since their difference would be bandlimited and also identically zero over a finite interval, a condition which violates the Paley–Wiener criterion. Thus in the analog case it is clear that the operators B and $(I - T)$ work together to insure that the iteration converges. Papoulis [9] has also shown convergence by alternative methods.

In the discrete case, the above argument does not hold. Indeed, it is possible to show that knowledge of a finite set of samples of a discrete bandlimited sequence as in (41) does not uniquely specify the sequence everywhere. In the Appendix it is shown that it is always possible to find a discrete bandlimited sequence which is identically zero over any finite interval of samples. Indeed there are an infinite number of such sequences having a given bandwidth. Since such a sequence could obviously be added to the original bandlimited sequence $x(n)$, leading therefore to the same sequence $y(n)$, it is clear that the result in the discrete case cannot be unique. Indeed it might be expected that this would lead to severe noise sensitivity problems with this algorithm.

Since the operator $G = (I - T)B$ is linear, the results of Section II-C apply in this case and closed-form expressions can be derived. Sabri and Steenart [10] have proposed a matrix formulation of the extrapolation problem based upon (22) and (23). Although such an approach is appealing, and indeed suggestive of the possibility of an exact solution, a word of caution is in order. The operation of bandlimiting produces a sequence of infinite duration. In implementing this operation using methods based upon the finite discrete Fourier transform, one must be aware of inherent 'wraparound' or aliasing effects that do not permit an exact solution except when the sequence x is a periodic sequence.

A similar approach to the extrapolation of a bandlimited signal was proposed by Cadzow [12]. This approach was based

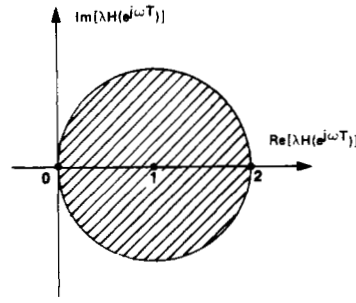

Fig. 1. Region in the complex $H(e^{j\omega T})$ plane for which convolution with $\delta - \lambda h$ is a contraction.

on the functional equation

$$x = (I - BT)\,x + By \tag{51}$$

which a first glance appears distinct from

$$x = y + (I - T)\,Bx \tag{52a}$$

upon which (43) is based. However, application of the band-limiting operator to both sides of (52a) gives

$$Bx = B(I - T)\,Bx + By = (I - BT)\,Bx + By. \tag{52b}$$

Because both B and I are linear operators, it can be seen that if an iteration scheme based upon (51) is started with a band-limited initial approximation By, as is shown to be necessary by Cadzow, it will lead to exactly the same set of iterates as the iteration scheme based upon (52a). However, as Cadzow points out [12] the order of operations implied by (51) may have advantages for discrete implementation.

It is perhaps important to note that the flexibility of choice of the basic functional equation can be exploited to advantage in obtaining convergent iterations. For example, if an operator F is nonexpansive but has a fixed point $x = Fx$, then the equation $x' = (1 - \alpha)\,x_0 + \alpha F x'$ will have a unique fixed point that can be obtained by successive substitutions if $0 \leqslant \alpha < 1$. It has been shown [13] that if the function space is a Hilbert space, then the $x' \to x$ as $\alpha \to 1$.

C. Linear Shift-Invariant Distortion

If the distortion operator is linear and shift-invariant so that

$$y = h * x \tag{53}$$

where h is the impulse response of the distorting system, then the restoration problem is commonly known as the deconvolution problem. Using this distortion model, we obtain from (12) the class of iterations for constrained deconvolution

$$x_0 = \lambda y \tag{54a}$$

$$x_{k+1} = \lambda y + q * Cx_k \tag{54b}$$

where $q = \delta - \lambda h$ and δ represents a unit impulse.

The conditions for convergence of this class of iterations can be obtained by considering for discrete signals

$$\|q * (\tilde{x}_i - \tilde{x}_j)\| = \left\{\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} |Q(e^{j\omega T})|^2 \cdot |\tilde{X}_i(e^{j\omega T}) - \tilde{X}_j(e^{j\omega T})|^2\, d\omega\right\}^{1/2} \tag{55}$$

where $\tilde{x}_i = Cx_i$ and $\tilde{x}_j = Cx_j$, and $\tilde{X}_i(e^{j\omega T})$ and $\tilde{X}_j(e^{j\omega T})$ are their respective Fourier transforms. If we replace $|Q(e^{j\omega T})|$ by its maximum value

$$r_1 = \max_{\omega} |Q(e^{j\omega T})| \tag{56}$$

then

$$\|q * (Cx_i - Cx_j)\| \leqslant r_1 \|Cx_i - Cx_j\|. \tag{57}$$

Now suppose that the constraint operator satisfies the condition

$$\|Cx_i - Cx_j\| \leqslant r_2 \cdot \|x_i - x_j\|. \tag{58}$$

Combining (57) and (58) leads to

$$\|q * (Cx_i - Cx_j)\| \leqslant r_1 r_2 \|x_i - x_j\|. \tag{59}$$

Clearly, the operator $G = (I - \lambda D)\,C$ will be a contraction if at least one of the operators $(I - \lambda D)$ and C is a contraction and the other is nonexpanding.

In the deconvolution case, a sufficient condition for $(I - \lambda D)$ to be a contraction is

$$|Q(e^{j\omega t})| = |1 - \lambda H(e^{j\omega T})| < 1, \qquad |\omega| < \pi/T. \tag{60}$$

This condition is depicted in Fig. 1, which shows that $\lambda H(e^{j\omega T})$ must be confined to the interior of a circular region of unit radius centered at unity. Assuming that λ is real and positive, it is clear from Fig. 1 that in order to satisfy (60), it is necessary that

$$\mathrm{Re}\,[H(e^{j\omega T})] > 0. \tag{61}$$

If (61) holds, it is clear that λ can be chosen so that (60) holds. For example, if $H(e^{j\omega T})$ is real and positive (implying that $h(n)$ has the properties of a discrete autocorrelation function) then it is easily seen that λ must satisfy

$$0 < \lambda < \frac{2}{\max_{\omega} [H(e^{j\omega T})]}. \tag{62}$$

To fully determine the requirements for convergence of the constrained deconvolution problem it is necessary to consider the constraint operator. At this point it is helpful to identify three special cases that have been discussed in the literature.

1) No Constraints: Perhaps the earliest reference to an iteration scheme of the type of (54) was by Van Cittert [14]. A discussion of this and other algorithms is given by Frieden [15]. In this case the constraint operator was the identity operator which is obviously nonexpansive.[1] Thus the conditions for convergence of the iteration are that $H(e^{j\omega T})$ satisfy (60) and (61) and that λ obey the conditions of (62) if $H(e^{j\omega T})$ is purely real. It should be noted that in this case, the operator

[1] A somewhat generalized version of the unconstrained iterative deconvolution algorithm was studied by Silverman and Pearson [16].

G is linear and shift-invariant and thus the discussion of Section II-C applies. By taking Fourier transforms in (54), and using the style of analysis of Section II-C, it can be shown that

$$X_k(e^{j\omega T}) = \lambda H_k(e^{j\omega T})\, Y(e^{j\omega T}) \tag{63}$$

where

$$\lambda H_k(e^{j\omega T}) = \frac{1 - [1 - \lambda H(e^{j\omega T})]^{k+1}}{H(e^{j\omega T})}. \tag{64}$$

Thus it is clear from (63) and (64) that if $H(e^{j\omega T})$ satisfies (60), then

$$\lim_{k \to \infty} \lambda H_k(e^{j\omega T}) = \frac{1}{H(e^{j\omega T})} \tag{65}$$

i.e., with no constraint, the iterative procedure converges to the same result as would be obtained with an inverse filter. Note that the condition that guarantees convergence of the procedure also guarantees that $|H(e^{j\omega T})| > 0$ so that the inverse filter exists for all ω.

In cases where (60) or (61) are not satisfied, convergence of the unconstrained iteration cannot be guaranteed. However, if a constraint operator is present, which is a contraction, it is only necessary that the operator $(I - \lambda D)$ (i.e., the impulse response $q = \delta - \lambda h$) be nonexpansive. In this case the condition on $H(e^{j\omega T})$ of (60) becomes

$$|1 - \lambda H(e^{j\omega T})| \leqslant 1, \quad |\omega| \leqslant \pi/T \tag{66}$$

so that $H(e^{j\omega T}) = 0$ is now permitted. This is an important point since it implies that with appropriate constraints it may be possible to restore frequency components that were lost by bandlimiting or other frequency selective filtering distortions. Thus it may be possible to deconvolve even in those cases where the inverse filter does not exist. Examples of such constraints are given below.

2) Finite Support Constraint: Suppose that it is known that x has finite support; i.e.,

$$x(n) \equiv 0, \quad n < n_a, \ n > n_b. \tag{67}$$

The appropriate constraint operator is the operator T which was defined in (41). The resulting iteration equations then become,

$$x_0 = \lambda y \tag{68a}$$

$$x_{k+1} = \lambda y + q * Tx_k \tag{68b}$$

where $q = \delta - \lambda h$. Iteration schemes of this form (with $\lambda = 1$) have been studied by Gerchberg [17] and Prost and Goutte [18].

If (60) holds, then a sufficient condition for convergence of the iteration of (68) is that T be nonexpansive. The iteration will still converge if (66) holds and T is a contraction. Because T is a linear operator, it is easily shown that

$$\|Tx_i - Tx_j\| = r_2 \|x_i - x_j\| \tag{69}$$

where

$$r_2 = \left\{ 1 - \frac{\sum_{n=n_a+1}^{\infty} |x_i(n) - x_j(n)|^2 + \sum_{n=-\infty}^{n_b-1} |x_i(n) - x_j(n)|^2}{\|x_i - x_j\|^2} \right\}^{1/2} \tag{70}$$

Clearly, $0 \leqslant r_2 \leqslant 1$, with $r_2 = 1$ only if x_i and x_j have finite support in $n_a \leqslant n \leqslant n_b$. Thus the iteration will converge if (60) holds and it may converge if (66) holds. The significance of the latter possibility is explored in more detail below.

3) Positivity Constraint: In many areas of practical interest, such as image processing and spectroscopy, it is sensible to impose a positivity (or nonnegativity) constraint. That is, it may be known that

$$x(n) \geqslant 0, \quad -\infty < n < \infty. \tag{71}$$

Such a constraint can be expressed by the positivity operator P defined in (5). Indeed it may be reasonable to impose both the positivity and finite support constraints [19]–[21]. To examine the effect of the positivity operator on convergence, we must consider

$$\|Px_i - Px_j\| = \left\{ \sum_{n=-\infty}^{\infty} |P[x_i(n)] - P[x_j(n)]|^2 \right\}^{1/2} \tag{72}$$

By considering four special cases (such as, for example, the case where both $x_i(n)$ and $x_j(n)$ are nonnegative) it is easily shown that

$$|P[x_i(n)] - P[x_j(n)]| \leqslant |x_i(n) - x_j(n)|. \tag{73}$$

Therefore, it follows that

$$\|Px_i - Px_j\| \leqslant r_2 \|x_i - x_j\| \tag{74}$$

where $0 \leqslant r_2 \leqslant 1$. It is clear in this case that $r_2 = 1$ only if x_i and x_j are both nonnegative. Thus as in the case of the finite support operator, the iteration scheme

$$x_0 = \lambda y \tag{75a}$$

$$x_{k+1} = \lambda y + q * Px_k \tag{75b}$$

will converge if (60) holds and it may converge if (66) holds.

The effect of either the finite support or positivity (or combined) constraints is particularly interesting when $h(n)$ is the impulse response of a bandlimited system. In this case (66) holds with equality for values of ω in the stopband of the system $(H(e^{j\omega T}))$, so that it is essential for the convergence of the iteration that the constraint operator be a contraction.

As an example, suppose that x is known to be positive and to have finite support. Then the appropriate iterative equation is

$$x_0 = \lambda y \tag{76a}$$

$$x_{k+1} = \lambda y + q * PTx_k \tag{76b}$$

where $q = \delta - \lambda h$. (Note that the operators P and T commute.) Now suppose that $h(n)$ is the impulse response of an ideal low-pass filter with frequency response

$$H(e^{j\omega T}) = \begin{cases} 1, & |\omega| < \Omega_c \\ 0, & \Omega_c < |\omega| \leqslant \pi/T. \end{cases} \tag{77}$$

Clearly, the contraction condition of (60) holds for $0 < \lambda < 2$ for $|\omega| < \Omega_c$, but (66) holds with equality for $\Omega_c < |\omega| \leqslant \pi/T$ no matter what value of λ is used. Thus, let us choose $\lambda = 1$ since it leads to a straightforward interpretation of the iterative relations expressed in (76).

In this case, the frequency response corresponding to $q(n)$ is the ideal high-pass filter with the frequency response

$$Q(e^{j\omega T}) = \begin{cases} 0, & |\omega| < \Omega_c \\ 1, & \Omega_c < |\omega| \leqslant \pi/T. \end{cases} \tag{78}$$

The initial approximation is simply $y = h * x$. It is clear that y will have infinite support even though x has finite support. Also even if x is positive, it is possible that y will be negative since $h(n)$ oscillates between positive and negative values. A simple example is a finite duration sequence x composed of a train of positive impulses, such as

$$x(n) = \sum_{m=1}^{M} \alpha_m \delta(n - n_m) \tag{79}$$

where $0 < \alpha_m$ for $m = 1, 2, \cdots, m$. The result of convolving this x with the impulse response of an ideal low-pass filter is neither of finite duration nor is it nonnegative. The first approximation is

$$x_1 = y + q * PTy. \tag{80}$$

Without the constraint operator PT, (i.e., in the Van Cittert case) nothing happens. To see this, note that $Q(e^{j\omega T})$ and $Y(e^{j\omega T})$ are disjoint, so that $x_1 = y$ and indeed $x_k = y$ for all k. If $H(e^{j\omega T})$ was bandlimited, but not perfectly flat in the passband, then the Van Cittert iteration would restore the passband, but not the higher frequencies. Thus as in the example of Section III-A, there is a built-in bandlimitedness that propagates from iteration to iteration. With the explicit constraints, however, it is easily seen that PTy will have energy beyond the cutoff frequency Ω_c. This energy is passed by the highpass filter q and, as seen in (80), these high frequencies are added to the low frequencies in y to produce the first approximation, x_1. The process is then repeated. Each iteration thus produces additional high-frequency components to add to the low frequencies provided by y.

The constraints thus offer the possibility of restoring high frequency information that has been lost in the linear shift-invariant distortion process. Note that x_k, the input to the constraint operator PT, will only satisfy the constraints in the limit, so that for i and j finite

$$\|PTx_i - PTx_j\| \leqslant r_2 \|x_i - x_j\| \tag{81}$$

with $0 < r_2 < 1$. Thus, if λ is chosen to satisfy (66), convergence to a solution satisfying $x = PTx$ and $y = h * x$ is guaranteed. Confirmation of the fact that the bandwidth can be extrapolated in a sensible way is provided by the examples of Section IV.

D. Constraints on a Signal and Its Fourier Transform

In a variety of problems (particularly in optics) the physics of signal generation implies certain constraints on the observed signal and its Fourier transform. For example, in optical systems it is possible to measure the magnitude of a signal (wavefront) and its Fourier transform, but measurement of the phase of either is very difficult. In such a situation, the distortion and constraints are intimately related. For example, consistency with a known magnitude is a constraint, while unknown phase can be thought of as a signal-dependent distortion, and vice versa. Gerchberg and Saxton [22] proposed an iterative algorithm for the reconstruction of a complex or bi-polar signal (image) from its magnitude and the magnitude of its Fourier transform. Fienup [23], [24] has considered iterative algorithms for reconstruction from the magnitude of the Fourier transform of a signal under the constraint that the signal is positive.

Hayes, Lim, and Oppenheim [25] have discussed algorithms for the reconstruction of signals with finite support from either the phase or the magnitude of the Fourier transform.[2] These and other similar iterative algorithms can be shown to fit into the general framework of Section II.

To illustrate this class of algorithms, suppose that either the magnitude or the phase of the Fourier transform of the signal x is known but not both. Also assume that prior knowledge of the properties of the signal can be expressed as a constraint operator; e.g., x may be known to be positive or to have finite support or both. The distortion can thus be represented in the Fourier domain as

$$Y = DX \tag{82}$$

where X and Y are the respective Fourier transforms of the desired signal x and the distorted signal y. That is, if only the magnitude of X is known, then the operator D could be defined by

$$Y = |X| = X \cdot e^{-j \arg[X]} \tag{83a}$$

or if only the phase is known

$$Y = e^{j \arg[X]} = X/|X|. \tag{83b}$$

Clearly the distortion is nonlinear and signal-dependent in both cases.

The constraint operator C can be most conveniently represented (and implemented) as the cascade of a Fourier-domain constraint operator, C_F, and a signal-domain operator C_S. The signal-domain constraint may be finite support [25] or positivity or whatever is appropriate. In the case where the magnitude of the Fourier transform is known (corresponding to the distortion of (83a)) then the Fourier-domain constraint is implemented as

$$V_k = C_F U_k = |X| \, e^{j \arg[U_k]} \tag{84a}$$

where $|X|$ is the known magnitude of the Fourier transform and U_k is the Fourier transform of the input to the Fourier-domain constraint operator. In the case where the phase is known, the Fourier-domain constraint operator would be implemented as

$$V_k = C_F U_k = |U_k| \, e^{j \arg[X]} \tag{84b}$$

where $\arg[X]$ is the known phase.

It turns out that the two different orderings of the Fourier-domain and signal-domain constraint operators lead to two different iteration equations. If $C = C_S \mathcal{F}^{-1} C_F \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform operator, then the general iteration equation can be represented in the Fourier-domain as

$$X_0 = \lambda Y \tag{85a}$$

$$X_{k+1} = \lambda Y + (I - \lambda D) \mathcal{F} C_S \mathcal{F}^{-1} C_F X_k \tag{85b}$$

where D is defined as in either (83a) or (83b). The operations of (85) are depicted in Fig. 2.

If the constraints are applied in reverse order, i.e., $C = \mathcal{F}^{-1} C_F \mathcal{F} C_S$, then the Fourier-domain iteration equation is

$$X_0 = \lambda Y \tag{86a}$$

$$X_{k+1} = \lambda Y + \tilde{X}_k - \lambda D \tilde{X}_k \tag{86b}$$

[2] It should be noted that such algorithms can also be thought of as signal *design* algorithms. For example, constraining the magnitude of the Fourier transform is equivalent to constraining the autocorrelation function of the signal. Thus the algorithms discussed in this section may be useful for designing signals with prescribed autocorrelation functions along with other prescribed properties.

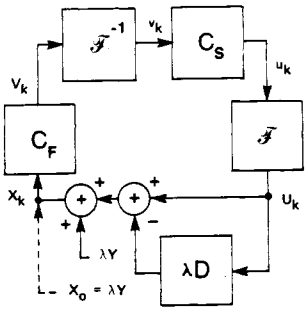

Fig. 2. Block diagram of an algorithm for constrained reconstruction from the magnitude or phase of the Fourier transform.

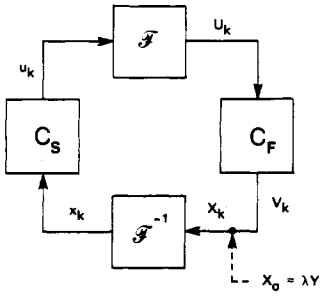

Fig. 3. Block diagram of a simpler algorithm for constrained reconstruction from the magnitude or phase of the Fourier transform.

where $\tilde{X}_k$ is the Fourier transform of the output of the constraint operator; i.e.,

$$\tilde{X}_k = C_F U_k \tag{86c}$$

where

$$U_k = \mathcal{F} C_S x_k. \tag{86d}$$

In both the known-magnitude and known-phase cases, no matter how the iteration is started, and independent of the iteration number, it can be seen that the term $\lambda D\tilde{X}_k$ in (86b) can be expressed as

$$\lambda D\tilde{X}_k = \lambda D C_F U_k = \lambda Y. \tag{87}$$

This is because the distortion and the Fourier-domain constraint are intimately related for both types of distortions. For example, if known-phase is the Fourier-domain constraint, then the combined operations DC_F will always produce a Fourier transform with unity magnitude and phase equal to the known phase.

Substituting the result of (87) into (86b) and expressing the iteration equation in terms of signal-domain quantities we obtain

$$x_0 = \lambda y = \lambda \mathcal{F}^{-1} D \mathcal{F} x \tag{88a}$$

$$x_{k+1} = C x_k = \mathcal{F}^{-1} C_F \mathcal{F} C_S x_k. \tag{88b}$$

The operations of (88) are depicted in Fig. 3. It is this simplified form of the phase (or magnitude) algorithm that is normally considered [23]–[25], since the other form, equation (85), clearly requires more computation per iteration.

In implementing either (85) or (88), we must implement the Fourier operators $\mathcal{F}$ and $\mathcal{F}^{-1}$. For discrete one-dimensional signals, these operators can be approximated adequately by finite discrete Fourier transforms [25]. Indeed, if the signal is known to have finite support, then it has been shown that it is theoretically possible to recover the signal exactly using the discrete Fourier transform [25].

It can be seen from (88) that convergence of the iteration will depend only upon the properties of the constraint operator. To demonstrate that the known-phase constraint coupled with a nonexpansive constraint such as finite support is nonexpansive, we consider

$$\|Cx_i - Cx_j\| = \|\mathcal{F}^{-1} C_F U_i - \mathcal{F}^{-1} C_F U_j\| \tag{89}$$

where $U_i = \mathcal{F} C_S x_i$ and $U_j = \mathcal{F} C_S x_j$. Applying the Fourier-domain constraint operator as in (88b) we obtain

$$\|Cx_i - Cx_j\| = \|\mathcal{F}^{-1} |U_i| e^{j \arg[X]} - \mathcal{F}^{-1} |U_j| e^{j \arg[X]}\|. \tag{90}$$

Assuming a Euclidean norm and using Parseval's theorem, the right-hand side of (90) can be expressed as

$$\|Cx_i - Cx_j\| = \left\{ \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \left| |U_i(e^{j\omega T})| - |U_j(e^{j\omega T})| \right|^2 d\omega \right\}^{1/2} \tag{91}$$

Using the triangle inequality and Parseval's theorem again we can show

$$\|Cx_i - Cx_j\| \leqslant \|\mathcal{F}^{-1} U_i - \mathcal{F}^{-1} U_j\| \tag{92}$$

with equality only if $\arg[U_i] = \arg[U_j] + 2\pi k$ for some integer k. Therefore,

$$\|Cx_i - Cx_j\| \leqslant r_1 \|C_S x_i - C_S x_j\| \tag{93}$$

where $0 \leqslant r_1 \leqslant 1$. Thus we have shown that C_F is nonexpansive. If either the positivity or finite support constraints are used for C_S, these are also nonexpansive, as we have already seen.

In problems of this type where constraints are imposed independently in both the signal domain and the Fourier domain, it is essential that these constraints be consistent and that there be a unique signal that satisfies both constraints. For example, consider the case of known phase with a finite support constraint in the signal domain. Clearly the set of signals whose Fourier transform has a given phase is infinite and likewise there are an infinite number of signals that are nonzero in a given region of support. The constraints are consistent if there is at least one signal that satisfies both constraints. On the other hand, the constraints may be consistent but they may not uniquely define the signal. In this example, if the known phase function has a linear component corresponding to one or more sets of four reciprocal zeros in its z-transform, then it will be impossible to reconstruct x from knowledge of the phase and finite support region alone since any set of four reciprocal zeros will produce the same linear phase component. This issue is treated in detail by Hayes, Lim, and Oppenheim [25] who give conditions for unique reconstruction from phase or magnitude. Their finding is basically that a unique reconstruction will occur if none of these reciprocal zeros are present in the signal.

IV. Implementation and Application of the Class of Constrained Deconvolution Algorithms

In the preceding sections we have discussed a general approach to constrained signal restoration, and we have de-

scribed several special cases of the general scheme. In this section, we shall consider the implementation and application of one of these special cases—the class of constrained discrete deconvolution iterations. We begin with a discussion of some details of the implementation of the iteration scheme. Then we describe the application of the method to the problem of improving the resolution of gamma ray spectra. We conclude with a discussion of convergence and the effects of noise. Although we specifically focus upon constrained deconvolution, it will be clear that many of the issues raised in this section also arise in the implementation and application of the other special cases discussed in Section III.

A. Implementation

The problem of recovering the input to a linear system from the system output and knowledge about the system (and possibly the input) arises in many situations. Generally an appropriate model is a continuous variable (analog) convolution of the form

$$y_a(t) = \int_{-\infty}^{\infty} x_a(\tau)\, h_a(t-\tau)\, d\tau \tag{94}$$

where $x_a(t)$ is the unknown input, $y_a(t)$ is the known output and $h_a(t)$ is the (known) impulse response of the linear system. Often the linear system passes only a finite bandwidth; i.e., the Fourier transform of $h_a(t)$ is bandlimited such that $H_a(\omega) \equiv 0$ for $|\omega| \geqslant \Omega$. Therefore, $y_a(t)$ can be sampled to obtain the sequence $y(n) = y_a(nT)$, and if $\pi/T \geqslant \Omega$, no aliasing will occur. Likewise, $h_a(t)$ can be sampled with the same period (also without aliasing) thereby obtaining the sequence $h(n) = h_a(nT)$. Under conditions of no aliasing we can easily show that the sequence of samples obtained by sampling $y_a(t)$ also satisfies the discrete convolution equation

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m) = x(n) * h(n) \tag{95}$$

where $h(n)$ is the sequence of samples of $h_a(t)$, and $x(n) = \tilde{x}_a(nT)$, where

$$\tilde{x}_a(t) = \int_{-\infty}^{\infty} x_a(\tau)\, \tilde{h}_a(t-\tau)\, d\tau \tag{96}$$

and $\tilde{h}_a(t)$ is the impulse response of a bandlimiting filter such that

$$\tilde{H}_a(\omega) = \begin{cases} 1, & |\omega| < \Omega \quad (97a) \\ \text{arbitrary}, & \Omega < |\omega| < \pi/T \quad (97b) \\ 0, & |\omega| > \pi/T. \quad (97c) \end{cases}$$

For example, $\tilde{h}_a(t)$ could be the impulse response of an ideal bandlimiting filter with bandwidth greater than or equal to the bandwidth Ω of the linear system $h_a(t)$. Clearly the solution of (95) for $x(n)$ cannot be unique, since $\tilde{h}_a$ is not unique. However, if we have prior knowledge of the properties of $x_a(t)$, we may be able to restore the signal frequencies beyond $\omega = \Omega$ by using the iterative schemes of (54) with an appropriate constraint operator. If we choose to apply the iterative deconvolution scheme to the discrete representation of (95), it is obvious that the best that we may hope for is to restore the desired signal in the band of frequencies $|\omega| < \pi/T$. This band can be made arbitrarily wide by choosing a high sampling rate (small sampling interval). However, practical considerations often dictate the use of the smallest possible sampling rate ($\pi/T = \Omega$) so as to minimize the number of samples that must be taken. In such cases it is necessary to increase the sampling rate after the initial sampling through the process of interpolation. This can be accomplished by discrete linear filtering; the details are given in [27]. Using these techniques, both the impulse response sequence h and the output sequence y would be interpolated using the same interpolation filter. If the interpolation filter has high attenuation at frequencies above $\omega = \Omega$, then the interpolated output can be represented accurately as

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m) \tag{97}$$

where

$$y(n) = y_a(nT) * h_i(n) \tag{98a}$$

$$h(n) = h_a(nT) * h_i(n) \tag{98b}$$

$$x(n) = \tilde{x}_a(nT). \tag{98c}$$

In (98), $h_i(n)$ is the impulse response of the interpolation filter and the new sampling interval is $T = MT'/L$ where T' is the original sampling interval. (Note: Interpolation is most convenient if M and L are integers [27].) It is interesting to note that for a nonideal interpolation filter $h_i(n)$ the sequence $y(n)$ will not be identical to the sequence $y_a(nT)$ that would be obtained by directly sampling $y_a(t)$ with period T. This is because of the extra filtering of the interpolation filter; however, this does not present a problem since both the output and the impulse response are filtered by the same filter. Thus, the interpolation filter need not have an extremely sharp cutoff since its effect in the band $|\omega| < \Omega$ will be removed by the deconvolution procedure

$$x_0(n) = \lambda y(n) \tag{99a}$$

$$x_{k+1}(n) = \lambda y(n) + q(n) * C[x_k(n)] \tag{99b}$$

where $q(n) = \delta(n) - \lambda h(n)$.

The implementation of (99) is seen to involve the discrete convolution of $q(n)$ with $C[x_k(n)]$. This convolution can be implemented directly, or when $q(n)$ and $C[(x_k(n)]$ are finite length sequences the convolution can be implemented by multiplication of discrete Fourier transforms [26]. In some cases, the bandlimited nature of the combined distorting system and interpolation filter may result in an impulse response that is very long, and thus rather large discrete Fourier transforms may be required to minimize wraparound effects in the discrete convolution.

An example of the performance of the iterative deconvolution algorithm on synthetic data is depicted in Fig. 4. The output sequence, $y(n)$, shown in Fig. 4(a) was obtained by convolving the input sequence

$$x(n) = \delta(n-30) + \delta(n-38) \tag{100}$$

with the impulse response

$$h(n) = Ae^{-(n^2/\sigma^2)} \tag{101}$$

where $A = 0.09974$, and $\sigma = 4$ samples. The magnitude of the Fourier transform of $y(n)$ is shown in Fig. 4(b). Note from the graph of $Y(e^{j\omega T})$ that $y(n)$ is sampled at about four times

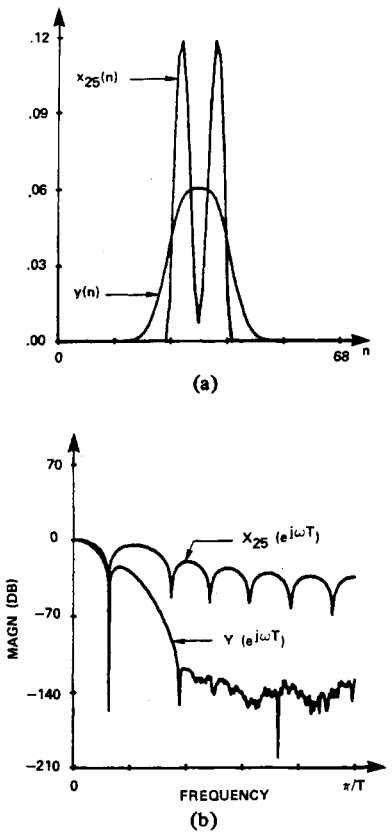

Fig. 4. Positive-constrained deconvolution of synthetic data. (a) Gaussian-blurred impulse pair $y(n)$ and the result $x_{25}(n)$ obtained after 25 iterations of (99). (b) Fourier transforms of the sequences in (a), showing the reconstruction of the high-frequency information.

the sampling rate needed to avoid aliasing. By using $h(n)$ given by (101) and $y(n)$ as shown in Fig. 4(a), in (99) with $\lambda = 2$ and the positivity constraint operator P of (5), the result $x_{25}(n)$ of Fig. 4(a) was obtained after 25 iterations. It is clear from Fig. 4(a) that a good approximation to $x(n)$ has been obtained and Fig. 4(b) shows that the high-frequency region has definitely been "filled in" by the iterative procedure. Carrying the iteration further leads to greater sharpness of the impulses or equivalently to a more faithful representation of the high frequencies. We have found that the high frequencies are reconstructed in a sensible way by the constrained iteration. By this we mean that the "arbitrary" region in (97b) is reconstructed in a way that is consistent with the constraints and the known distortion operator.

A second example is shown in Fig. 5. In this case, the two-dimensional Gaussian sequence of Fig. 5(a)

$$h(m,n) = \exp\left[-\left(\frac{m^2+n^2}{100}\right)\right] \tag{102}$$

is convolved with the sequence

$$x(m,n) = [\delta(m-24)+\delta(m-34)]\,\delta(n-32) \tag{103}$$

to produce the two-dimensional sequence $y(m,n)$ of Fig. 5(b). The iterative equation in two-dimensional form is

$$x_0(m,n) = \lambda y(m,n) \tag{104a}$$

$$x_{k+1}(m,n) = \lambda y(m,n) + q(m,n) ** C[x_k(m,n)] \tag{104b}$$

where $q(m,n) = \delta(m)\,\delta(n) - \lambda h(m,n)$, ** denotes two-dimensional discrete convolution, and $C[\ ]$ denotes the combined positivity and finite support operator. The result of 65 iterations is shown in Fig. 5(c).

We should recall that (99) and (104) represent a solution to the discrete deconvolution problem. A fundamental question arises when we attempt to relate the solution $x(n)$ to the underlying analog signal $x_a(t)$. In the case of the example of Fig. 4 and for the gamma-ray spectra to be discussed next, it is reasonable to assume an underlying analog model with

$$x_a(t) = \sum_{k=1}^{N} a_k \delta_a(t-\tau_k) \tag{105}$$

where $\delta_a(t)$ is the impulse function and the areas of the impulses (the parameters $\{a_k\}$) are positive. This signal clearly satisfies both the finite support and positivity constraints. Only under very special circumstances, however, will it be possible to obtain a sequence of samples of a bandlimited version of $x_a(t)$ that will satisfy both of these constraints. This can be seen by considering the ideal bandlimited version

$$\tilde{x}_a(t) = \sum_{k=1}^{N} a_k \frac{\sin[\Omega(t-\tau_k)]}{\Omega(t-\tau_k)}. \tag{106}$$

It is easily seen that if $\Omega = \pi/T$ and if $\tau_k = n_k T$ where n_k is an integer, then

$$x(n) = \sum_{k=1}^{N} a_k \delta(n-n_k). \tag{107}$$

That is, $x(n)$ is a finite length sequence of positive discrete impulses. If any of the analog impulses do not fall exactly on the sample points, the sequence $x(n)$ will be neither positive nor of finite length. However, if $x_a(t)$ is filtered by a positive impulse response with finite support ($\tilde{h}_a(t) \geqslant 0$), then

$$x(n) = \tilde{x}_a(nT) = \sum_{k=1}^{N} a_k \tilde{h}_a(nT-\tau_k) \tag{108}$$

will be a positive sequence with finite support. Of course, $\tilde{h}_a(t)$ can not be bandlimited and also have finite support. Since the bandlimitedness is a built-in assumption of the discrete iterative reconstruction algorithm we see that there is a fundamental limitation on the accuracy with which we can reconstruct the analog signal $x_a(t)$ using the discrete algorithm. Thus although the iterative algorithm can give an exact solution to the discrete deconvolution problem, we must in general interpret the solution as samples of an approximation to $x_a(t)$. In a practical sense, we will be content if the discrete solution retains the basic structure of the underlying analog signal. In the case of (102) this means that we should be able to get accurate estimates of the parameters a_k and τ_k. The application of the constrained iterative algorithm to a sampled signal is our next concern.

B. Application to Gamma-Ray Spectra

Gamma-ray spectra are typical of many types of physical measurements where a line (or impulse) spectrum is blurred by the finite resolution of the measurement instrument. A reasonable model for such spectra (signals) is the convolution of an impulse train function as in (105) with $a_k > 0$ and a positive blurring function, often of Gaussian shape. An example

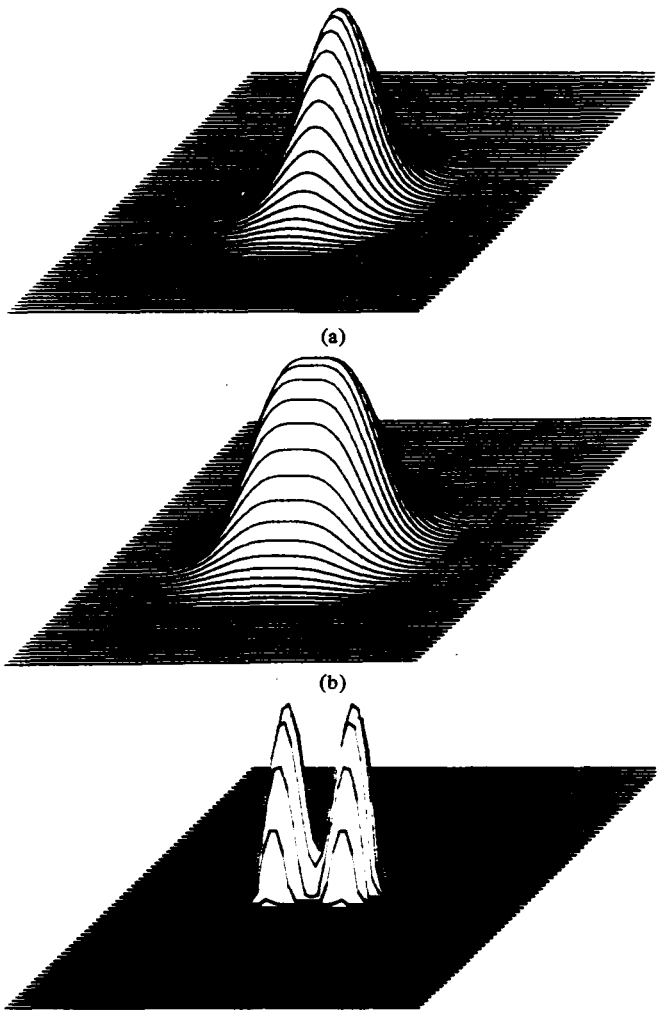

Fig. 5. Two-dimensional positive-constrained deconvolution of synthetic data. (a) Gaussian blurring function $h(m, n)$. (b) The sequence $y(m, n)$ obtained by convolving $h(m, n)$ with an impulse pair. (c) Estimate of the impulse pair obtained after 65 iterations with $\lambda = 2$.

of a portion of the gamma-ray spectrum of ^{177m}Lu is shown in Fig. 6. Since a gamma-ray spectrum is a plot of photon count versus frequency, the index n is proportional to energy ($E = h\nu$). As is readily seen in Fig. 6, this model appears to be appropriate except for a slowly varying background radiation. In processing this signal, we have removed the background by fitting a low-order polynomial through selected points of the signal assumed to represent background radiation only and then subtracting that polynomial before processing. Fig. 7(a) shows the segment labelled "analysis segment" in Fig. 6 after removal of the background and interpolation by 4:1. This is the blurred signal $y(n)$. Fig. 7(b) shows the isolated spectral line indicated in Fig. 6 after background removal and 4:1 interpolation. This isolated blurred spectral line provides our estimate of $h(n)$. The remaining parts of Fig. 7 show various methods of restoration applied to $y(n)$.

Fig. 7(c) shows the output of an inverse filter implemented by dividing the discrete Fourier transform of $y(n)$ by the discrete Fourier transform of $h(n)$. The transforms were computed with 1024 points to avoid wrap-around effects and the quotient of transforms was set to zero for frequencies $\omega \geqslant 0.25(\pi/T)$ where the Fourier transform of $h(n)$ becomes too

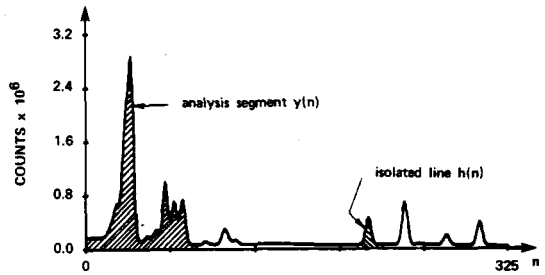

Fig. 6. A portion of the gamma-ray spectrum of ^{177m}LU. The shaded segments are expanded in Fig. 7.

small. Note that the abrupt bandlimiting causes large oscillations that violate the inherent positivity of gamma-ray spectra and tend to obliterate the low-level spectral lines.

Fig. 7(d) shows the result of 20 iterations of the iterative algorithm with no constraints and $\lambda = 1$ (Van Cittert's case). Note that the oscillations are still in evidence although they are less objectionable than in Fig. 7(c). Recall, however, that if a large number of iterations were used, the result would be the same as the inverse filter result of Fig. 7(c).

Fig. 7(e) shows the result of 20 iterations with the finite support constraint imposed and $\lambda = 1$. In this case, the region of support can be estimated easily from $y(n)$ since $y(n)$ is essentially zero outside a clearly defined interval. It can be seen that, in this case, the constraint does not produce a noticable improvement in resolution and, as expected, it has no effect on the undesirable oscillations. The reason that the finite support constraint has little effect can be found in the discussion of Section III-C1. There the constraint operator is directly responsible for generating the restored high frequency components. In this case the assumed finite support region is so wide that its truncating effect is negligible on the signals generated by the iterative procedure. Thus the constraint operator C does not function well as a contraction, and little high frequency energy is restored.

Fig. 7(f) shows the result of 20 iterations with both the finite support and positivity constraints imposed and $\lambda = 2$. Here the effect of the positivity constraint is dramatically in evidence. The resulting output has greatly enhanced resolution with no objectionable artifacts.

C. Convergence

In Section III-C it was shown that a condition for convergence of the iterative deconvolution algorithm of (99) is that

$$|1 - \lambda H(e^{j\omega T})| \leqslant 1, \qquad |\omega| \leqslant \pi/T \tag{109}$$

provided that the constraint operator is a contraction. Assuming that λ is a positive real constant, equation (109) implies, in turn, that

$$\mathrm{Re}\,[H(e^{j\omega T})] \geqslant 0, \qquad |\omega| < \pi/T. \tag{110}$$

In considering these conditions for convergence, we must first check to see if (110) is satisfied. If it is satisfied, it will always be possible to choose a value for λ that will ensure that (109) is satisfied. Otherwise no choice of λ can insure convergence. If (110) is not satisfied, it may still be possible to use the iterative approach, as we shall see, but before discussing the required modifications let us first consider the choice of λ when (110) is satisfied.

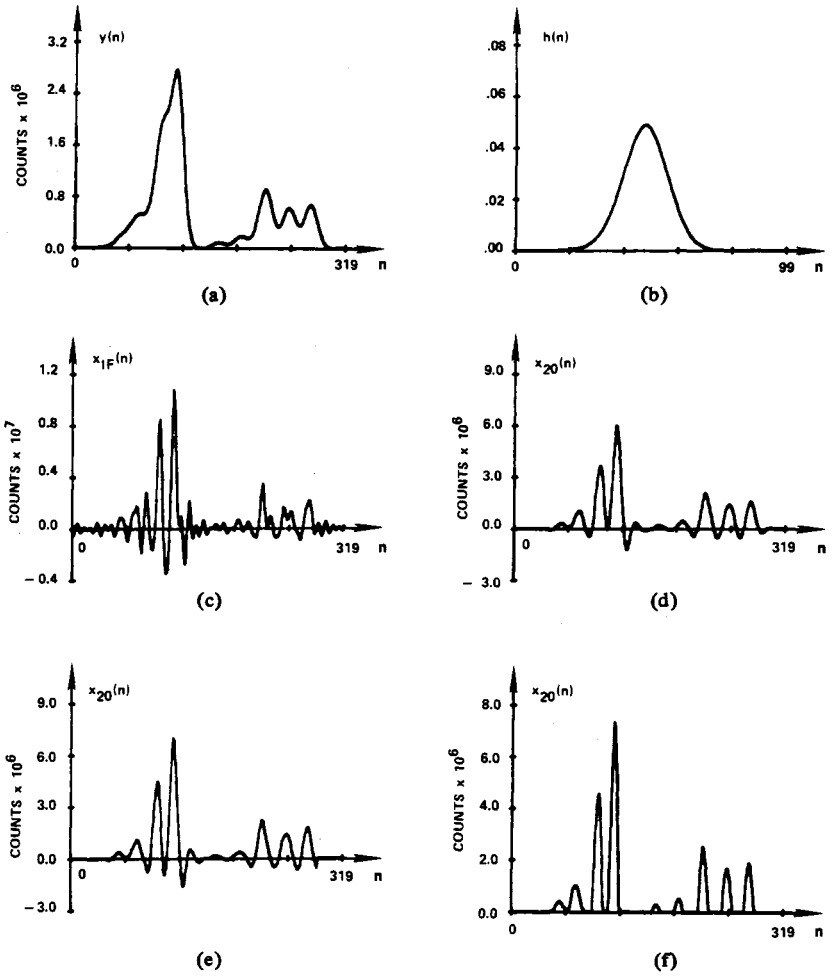

Fig. 7. Comparison of deconvolution schemes on gamma-ray spectrum data. (a) The left shaded portion of the data in Fig. 6 after background removal and 4:1 interpolation. This is the sequence $y(n)$. (b) The estimate of $h(n)$ obtained from the right shaded portion of Fig. 6 after background removal, interpolation, and normalization. (c) Output of the inverse filter with cutoff frequency $\omega_c = (0.25)(\pi/T)$. (d) Result obtained from the iterative algorithm with $\lambda = 1$ and no constraints (Van Cittert's algorithm). (e) Result obtained with $\lambda = 1$ and a finite support constraint. The region of support is $45 \leqslant n \leqslant 290$. (f) Result obtained with $\lambda = 2$ and both finite support and positivity constraints. The sequences (d)-(f) were all obtained after 20 iterations.

In the previous examples of Sections IV-A and B, the impulse response of the blurring system was either a perfectly symmetric Gaussian pulse with positive real Fourier transform, or, in the case of the gamma-ray spectra, the estimate of the impulse response was an isolated spectral line with very much the same property. In applying the iterative algorithm, $h(n)$ was normalized so that

$$H(e^{j0}) = \sum_h h(n) = 1. \tag{111}$$

This ensures that the "areas" of the deconvolved "impulses" are correct. Since $H(e^{j\omega T})$ has a maximum value of 1 at $\omega = 0$, it follows from (62) that $0 \leqslant \lambda \leqslant 2$ for convergence. The value of λ in this range that maximizes the rate of convergence is the value that minimizes the maximum value of

$$|Q(e^{j\omega T})| = |1 - \lambda H(e^{j\omega T})|, \qquad |\omega| \leqslant \pi/T \tag{112}$$

as can be seen from (56). If $H(e^{j\omega T})$ is complex, finding the optimal value for λ may be rather complicated but for a low-pass system with $H(e^{j\omega T})$ real and nonnegative as in the previous examples and normalized according to (111), it is easily shown that the optimal choice is $\lambda = 2$.

To illustrate the effect of λ on convergence rate for signals such as those in Sections IV-A and IV-B, the Gaussian blurring impulse response of (101)

$$h(n) = 0.09974e^{-(n/4)^2} \tag{113}$$

was convolved with a unit sample so that $y(n) = h(n)$ in the deconvolution iteration. The resulting outputs become sharper with each successive iteration as depicted in Fig. 8. The width of the output pulses is defined as the width at half the maximum value and denoted Δ_k. In Fig. 9 the quantity Δ_k/Δ_0 is plotted as a function of k for various conditions. The curve labeled A (dashed) shows Δ_k/Δ_0 versus k for $\lambda = 1$, while curve B is for $\lambda = 2$. Note that these curves display the type of geometric convergence that is typical of iterations of this type.

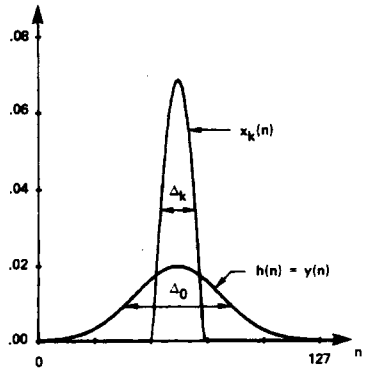

Fig. 8. Illustration of the measurement of the width of positive pulses. The width Δ is taken at one half of the peak of the pulse.

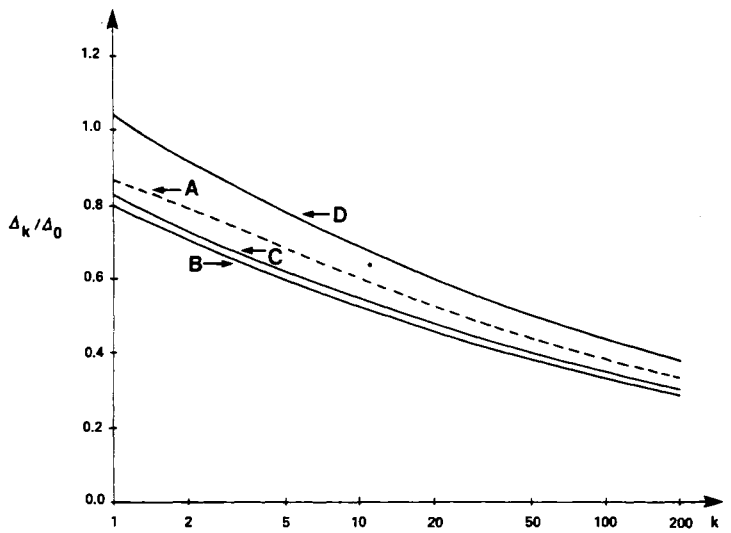

Fig. 9. Convergence of the deconvolution algorithm with finite support and positivity constraints. This graph plots the normalized width of a single blurred impulse versus the number of iterations for various choices of λ and prefilter cutoff frequency. The cutoff frequency of the prefilter is $\omega_c = (BWF)(\pi/T)$. (a) $\lambda = 1$, $BWF = 1.0$. (b) $\lambda = 2$, $BWF \geqslant 0.5$. (c) $\lambda = 2, BWF = 0.25$. (d) $\lambda = 2, BWF = 0.1$.

It can be seen that to achieve a given normalized width requires about twice as many iterations with $\lambda = 1$ as with $\lambda = 2$. Since the iterative process is very computationally intensive, this is a significant improvement.[3]

Let us now return to the question of what can be done when $H(e^{j\omega T})$ does not have a nonnegative real part. A simple approach is to filter both the distorted signal $y(n)$ and the distortion impulse response $h(n)$ with an impulse response $h_c(n)$, such that

$$\mathrm{Re}\,[H(e^{j\omega T})\,H_c(e^{j\omega T})] \geqslant 0. \tag{114}$$

If such a compensating filter can be found, then it will be possible to find a λ such that (109) is satisfied with $H(e^{j\omega T})$ replaced by $H(e^{j\omega T})\,H_c(e^{j\omega T})$. There are many compensating systems that will work; however, one that will always work is

$$h_c(n) = h^*(-n) \tag{115}$$

where * denotes complex conjugation. Since for this case

$$H(e^{j\omega T})\,H_c(e^{j\omega T}) = |H(e^{j\omega T})|^2 \geqslant 0. \tag{116}$$

This choice also has the advantage that the resulting frequency response of the combined system is real so that it is easy to see from (112) that the optimal choice of λ is

$$\lambda = \frac{2}{\max\limits_{\omega} |H(e^{j\omega T})|^2}. \tag{117}$$

It should be pointed out that in order to implement the convolution of $h_c(n) = h(-n)$ with $y(n)$ and $h(n)$, it must be true that $h(n)$ has finite support. In practice this may not be a serious limitation because of the natural tendency for $h(n)$ to decay with increasing n.

To illustrate this approach to insuring convergence, consider an impulse response of the form

$$h(n) = \begin{cases} 1/(2M+1), & |n| \leqslant M \\ 0, & \text{otherwise.} \end{cases} \tag{118}$$

[3] Jansson *et al.* [28] have considered ways of varying λ in a signal dependent way so as to speed up convergence.

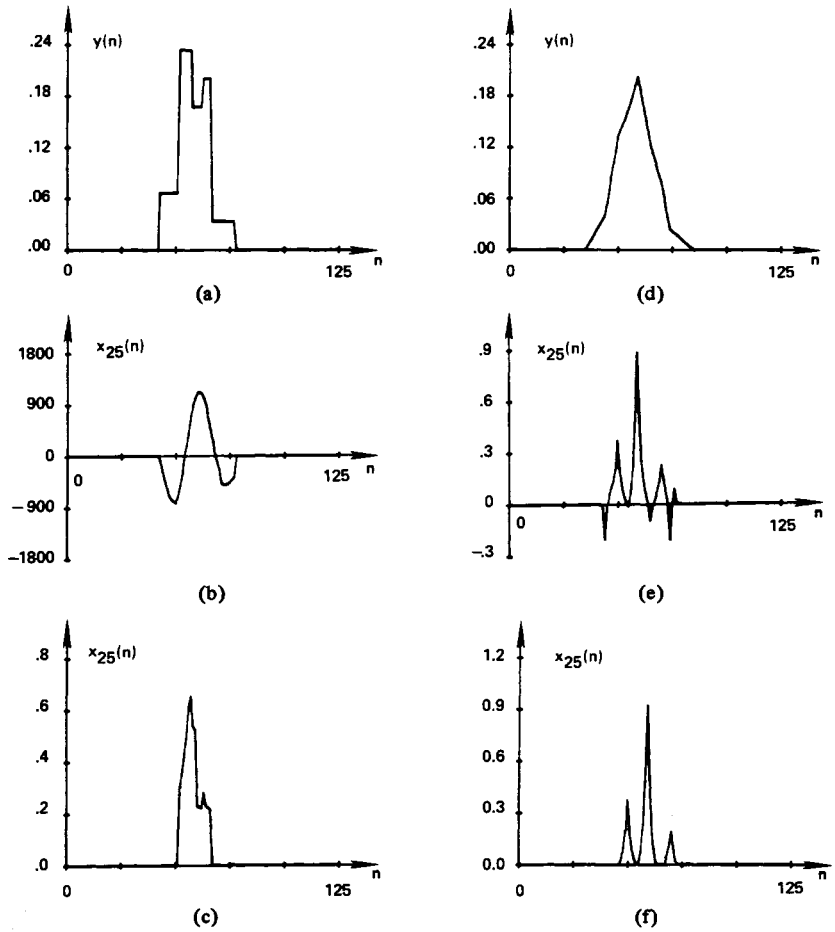

Fig. 10. Illustration of compensation to insure convergence. (a) The input sequence $y(n)$ of (120). (b) Result obtained with a finite support constraint. Note the unstable growth of the oscillation. (c) Result obtained with finite support and positivity constraints. (d) The new input $y(n)$ after convolution with $h_c(n)$. (e) Result obtained from the compensated system with finite support constraint. (f) Result obtained with finite support and positivity constraints. The sequences in (b), (c), (e), and (f) were all obtained after 25 iterations with $\lambda = 2$. The region of support was restricted to $45 \leq n \leq 77$.

The corresponding Fourier transform is

$$H(e^{j\omega T}) = \frac{\sin\,[(2M+1)\,\omega T/2]}{\sin\,[\omega/T]} \tag{119}$$

which does not satisfy the contraction constraint in (110). This impulse response (or its analog counterpart) is a useful model for many physical distortions; e.g., motion blur in images. Fig. 10(a) shows the result of convolving $h(n)$ of (118) ($M = 7$) with an impulse train to produce the sequence

$$y(n) = h(n-50) + 2.5h(n-59) + 0.5h(n-70). \tag{120}$$

The result of using the iterative algorithm with $\lambda = 2$ and the finite support constraint is shown in Fig. 10(b), and with the positivity constraint in Fig. 10(c). Note the unstable oscillatory behavior in Fig. 10(b) and note that even with the positivity constraint the algorithm does not distinguish the three impulses at all. If $y(n)$ and $h(n)$ are convolved with $h_c(n) = h(-n)$, the resulting sequence $y(n) * h_c(n)$ is as shown in Fig. 10(d) and the resulting deconvolved outputs with finite support and positivity constraints are shown in Figs. 10(e) and 10(f), respectively. In this case the peaks are at the correct locations and the heights are in the correct proportion. Thus it appears that the compensation technique is effective in removing a fundamental limitation of the iterative deconvolution algorithm.

D. Effects of Additive Noise

The previous examples have demonstrated the effectiveness of the iterative deconvolution algorithm when the signal is noiseless; i.e., the convolutional model is accurate. Even if the convolutional model accurately fits the physics of signal generation, it is likely that there will be a random error or disturbance in the measurement of recording of the signal. It is therefore important to assess the effects of such noise and to attempt to mitigate the deleterious effects that this noise has on the deconvolution process.

To illustrate these effects let us assume that the sequence $y(n)$ is corrupted by additive broad-band random noise, but that a relatively noiseless estimate of $h(n)$ is available. Such an estimate is often available through auxiliary measurements. Furthermore, in many applications $h(n)$ is approximately Gaussian in shape, and a Gaussian substitute may be used with little degradation of the results.

The addition of even moderate amounts of noise to the data

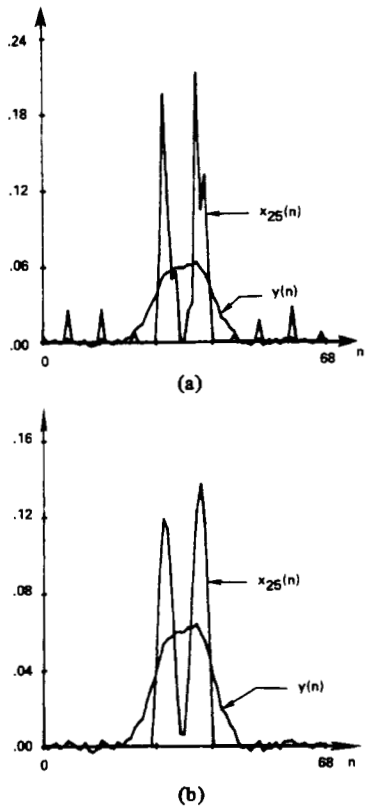

Fig. 11. Prefiltering for reconstruction from noisy data. (a) Positive constrained deconvolution of the data of Fig. 4(a) with noise added. The result $x_{25}(n)$ shows a tendency to split the two correct peaks and 7 small spurious peaks. (b) Result obtained for $x_{25}(n)$ after prefiltering $y(n)$ and $h(n)$ with a lowpass filter of cutoff frequency $\omega_c = (0.25)(\pi/T)$. The artifacts are suppressed. Compare with Fig. 4(a).

has serious effects on the result. For blurred impulsive data, errors in the estimate of $x(n)$ include splitting of single peaks into multiple peaks, the development of spurious peaks, and the obscuration of low-level peaks. These effects are illustrated in Fig. 11. Fig. 11(a) shows the signal from Fig. 4(a) corrupted by uniform pseudo-random noise with a signal-to-noise ratio (SNR) of 30 dB (SNR is here defined as the ratio of the square of the maximum of the noiseless signal to the noise variance), along with the result after 25 iterations of the deconvolution algorithm. This result shows both the splitting of the two large impulses and the development of several small spurious impulses.

Due to the low-pass nature of $h(n)$ and thus of $y(n)$, the ratio of the signal power spectrum to the noise power spectrum is generally higher for low frequencies than for high frequencies. Since the deconvolution procedure tends to seek a solution by extrapolating from known spectral segments of $x(n)$ to unknown segments, it might be possible to improve the result in the noisy case if the iteration were based only on the data from the lower frequency regions where the signal power is greatest. This observation leads to a simple method for combatting the effects of noise on the algorithm.

The modification to the algorithm consists of low-pass filtering both the data $y(n)$ and the blurring function $h(n)$ prior to applying the iterative procedure, as in the case of compensation to insure convergence. The effect again is to solve a substitute problem described by the relation

$$y'(n) = h_c(n) * y(n) = h'(n) * x(n) \tag{121}$$

where the new effective blurring function $h'(n)$ is given by

$$h'(n) = h_c(n) * h(n) \tag{122}$$

and $h_c(n)$ is the unit-sample response of the low-pass filter applied to $y(n)$ and $h(n)$. The sequences $y'(n)$ and $h'(n)$ are now used in place of $y(n)$ and $h(n)$ in the procedure. However, whereas $y(n)$ contains broad-band noise, $y'(n)$ contains relatively narrowband noise of less total power, much of the noise having been removed by filtering. This idea is easily extended to deal with bandpass noise or more general blurring functions. Its efficacy is a function of the concentration in frequency of the noiseless signal $y(n)$. An additional benefit of this procedure derives from the substitution of $h'(n)$ for $h(n)$. If the estimate of $h(n)$ is corrupted by broad-band noise, then this noise too will be reduced by the filtering process. It should be pointed out that we can only expect to achieve improved performance in the constrained case, since we rely on the constraints to regenerate the high frequencies removed by the filtering.

The implementation of this modification adds very little to the computation involved in deconvolution. The filtering of $y(n)$ and $h(n)$ is required only once prior to initiating the iteration. If interpolation of the data is also required, as discussed in Section IV-A, the two filtering operations may be combined into one. If compensation with $h'(-n)$ is required to gain convergence this also can be done at the same time. Note that such compensation, however, must be based upon the new impulse response, i.e., the impulse response which includes the contribution of any noise reducing filters.

The effect of this modification to the algorithm is illustrated in Fig. 11(b) which continues the example of Fig. 11(a). This shows the result obtained after 25 iterations when the noisy data was prefiltered with a low-pass filter with a cutoff frequency expressed as a fraction of the total possible bandwidth of the data. Thus $BWF = 0.25$ means that the cutoff frequency of $h_c(n)$ is $\omega_c = (0.25)(\pi/T)$.

In general as the passband of the low-pass filter is made narrower, a greater degree of noise rejection is obtained. At the same time, however, the algorithm is provided with information about $x(n)$ over a more limited portion of the spectrum so that a greater degree of bandwidth extrapolation is required. When the filtering becomes so severe as to remove spectral regions in which $y(n)$ has significant energy, increasing numbers of iterations are required to maintain a given resolution in the result. Thus, the improved performance on noisy data is obtained at the cost of greater computation.

As a further example, Fig. 12(a) shows a noisy segment from a measured gamma-ray spectrum and the resultant output after 25 iterations of the deconvolution algorithm with $\lambda = 2$ and the positivity constraint. Fig. 12(b) shows the input and output for $BWF = 0.25$. Figure 12(b) is clearly the more reasonable result.

The tradeoff between noise rejection and resolution is shown by curves B, C, and D in Fig. 9. Recall from the discussion of Section IV-C that curve B shows the dependence of the width of a single deconvolved Gaussian pulse on the number of iterations for $\lambda = 2$. Curve B holds for all values of BWF between 0.5 and 1.0. Curve C is for $BWF = 0.25$ and D is for $BWF = 0.1$. Clearly, restricting the bandwidth slows the rate of convergence, but not dramatically. For example restricting the bandwidth to only 25 percent of the original band increases the number of iterations required for a given width by only a

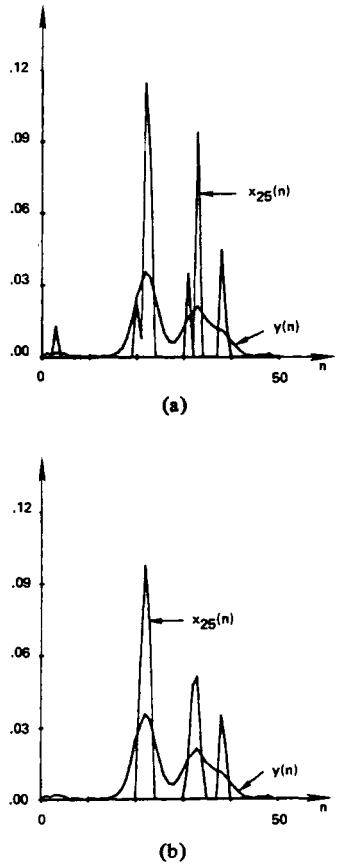

Fig. 12. Positive-constrained deconvolution of a segment of a noisy gamma-ray spectrum for ^{201}mPO. (a) Without prefiltering. (b) With prefiltering. The cutoff frequency is $\omega_c = (0.25)(\pi/T)$. Both results were obtained after 25 iterations with $\lambda = 2$.

factor of 1.25, while using only 10 percent of the original band increases the number of iterations by a factor of 4.

Thus the simple low-pass filtering operation is effective in mitigating the effects of additive noise. A possible refinement of this approach would be to use an optimum smoothing filter of the Wiener type instead of the low-pass filter. In any case it is interesting to note that as in the case of the interpolation and convergence compensation operators we have shown that by introducing additional linear distortion it is possible to improve the performance of the iterative deconvolution algorithm.

E. Shift-Varying Blur—An Example

In many practical applications, the appropriate distortion model is shift-varying. This is in fact the case for gamma-ray spectra, where the amount of blurring (width of the observed spectral lines) increases with energy. In such cases, where the distortion is linear but shift-varying, the distortion operator is described by the superposition sum

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n, m) \tag{123}$$

where $h(n, m)$ is the response of the system to a unit sample at index m. As we have pointed out, the iterative restoration approach can be applied in the shift-variant case, but of course the two-dimensional sequence $h(n, m)$ must be known. In general, complete knowledge of $h(n, m)$ may be impossible to obtain; however, in cases of spectrum line broadening where the line shape is fixed, it may be possible to estimate the variation of linewidth with the independent variable. In such cases an appropriate model might be

$$h(n, m) = \exp\left[-((n - m)/\sigma(m))^2\right]. \tag{124}$$

That is, the response of the distortion system to an impulse $\delta(n - m)$ is a Gaussian pulse with standard deviation $\sigma(m)$ centered at $n = m$. In this case only $\sigma(m)$ need be known in order to implement the shift varying operator. ($h(n, m)$ would of course have to be limited to a finite region of support in order to carry out the operation of (123)).

The following simulated example, due to Marucci [29], illustrates the value of the iterative approach. In this example, the true input $x(n)$ is the uniform impulse train

$$x(n) = \sum_{k=0}^{15} \delta(n - 50 - 10k) \tag{125}$$

as depicted in Fig. 13(a). The shift-varying blurring function used was that of (124) with

$$\sigma(m) = 2 + (m - 50)/100. \tag{126}$$

The blurred signal $y(n)$ is shown in Fig. 13(b).

First, $y(n)$ was processed using the iterative algorithm with positivity and finite support constraints and assuming that the distortion was *shift-invariant* with impulse response

$$h(n) = e^{-n^2/4}. \tag{127}$$

The result after 500 iterations is shown in Fig. 13(c). Note that the first four or five impulses are well restored; however, the later impulses where the blurring was much greater than that of (127), are not recovered. When the true distortion operator specified by (123), (124), and (126) was used in the iterative algorithm with the same constraints, the result after 500 iterations was as shown in Fig. 13(d). Clearly this result is far better than in the shift-invariant one. Notice that the later impulses where the degree of blur was the greatest were not completely restored after 500 iterations. This is consistent with the results of Section IV-B where it was shown that the width of the restored Gaussian pulse decreases roughly as the logarithm of the iteration number. Thus a far greater number of iterations will be required to achieve a given width for the later impulses than for the earlier ones.

The results obtained with this simulated example are an indication that iterative restoration may be possible in certain cases of shift-varying distortion. When sufficient knowledge about the distorting system is available, the iterative approach may permit restoration with constraints, without the need to implement a shift-varying inverse system and without resorting to locally shift-invariant approximations.

V. Summary

This paper has described a rather broad class of iterative signal restoration techniques. We have shown that the basic functional equation

$$x = Cx + \lambda(y - DCx) \tag{128}$$

can be applied with many different types of distortions and constraints, and we have shown that under certain conditions the method of successive substitutions leads to a convergent iterative solution. It should be reemphasized that the formula-

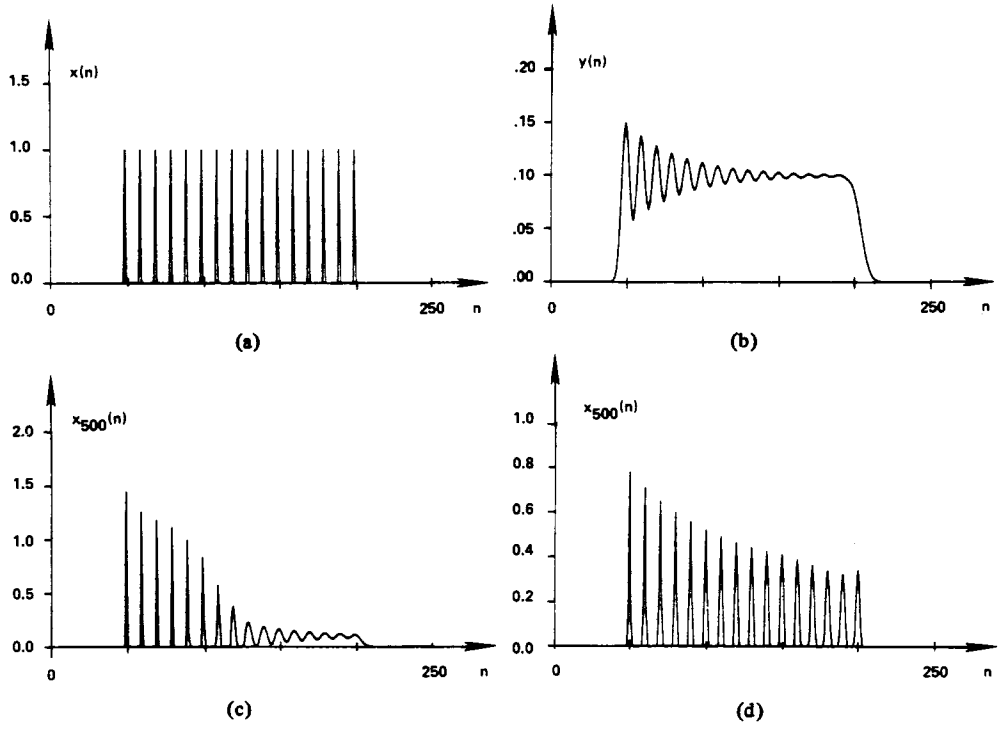

Fig. 13. Positive-constrained reconstruction from a shift-varying blur. (a) Original impulse train $x(n)$. (b) Blurred sequence $y(n)$. (c) Result obtained after 500 iterations assuming a shift-invariant blur. (d) Result obtained after 500 iterations using the correct shift-varying blur.

tion explored in detail here is not unique and that other basic functional equations may have advantages for certain distortions and constraints.

In addition to discussing the general framework and surveying a number of iterative techniques, we have elaborated upon the details of a class of iterative deconvolution algorithms and their application to blurred impulsive signals. In Section IV it was shown that the constrained deconvolution algorithms are sensitive to additive noise, but that the effects of noise can be mitigated by a judicious combination of predistortion for filtering the noise and constraints for restoring signal information that has been lost. It appears that this approach could be useful with other distortion and constraint operators; however, it should be emphasized that we have not given a general procedure for doing this. It seems clear, however, that the basic goal should be to make the available signal better fit the model imposed by the combined distortion and constraint operators. Clearly, further work is required to establish a general approach to finding constraint operators (and predistortion operators) that insure convergence to solutions consistent with the underlying physics of the problem.

In addition to the new insight into specific algorithms and their inter-relationships, it is hoped that the general principles established herein may lead to new algorithms for new combinations of distortions and constraints. An example is the illustration of Section IV-E. In the case of shift-variant distortions, a major advantage is the fact that only the distortion operator need be implemented rather than its shift-varying inverse. Another example, which we have just begun to study, is the case where we are given multiple distorted versions $y_i = \boldsymbol{D}_i x$, $i = 1, \cdots, N$, of the desired signal x. Following the same approach as before we easily arrive at the iteration equation

$$x_{k+1} = x_0 + \boldsymbol{G} x_k \tag{129}$$

where

$$x_0 = \sum_{i=1}^{N} \lambda_i y_i \tag{130}$$

and

$$\boldsymbol{G} = \left(I - \sum_{i=1}^{N} \lambda_i \boldsymbol{D}_i \right) \boldsymbol{C}. \tag{131}$$

As before $\boldsymbol{C}$ is an appropriate constraint operator. A notable example of a problem where such an algorithm could be applied is the reconstruction of multidimensional signals from projections [30]. In this case, the $\boldsymbol{D}_i$ would be (linear) projection operators and $\boldsymbol{C}$ might be a positivity constraint.

A final comment should be made about convergence of iterative schemes. It should be emphasized that the underlying functional equation is what is being solved, and the method of successive substitutions is not the only iterative approach that can be followed. In many cases it may, in fact, be the least efficient approach. Other schemes, which may involve more computations per iteration, may converge in many fewer iterations [29]. The application of these more sophisticated solution methods may lead to wider applicability of this approach to signal restoration—particularly in the case of multidimensional signals. In any case, the clear advantage of this approach and the increasing availability of high-speed digital processors suggest that the techniques that we have described for solution of signal restoration problems are worthy of consideration in many application areas.

Appendix
A Property of Discrete Bandlimited Sequences

Consider a discrete bandlimited sequence $V(n)$ such that its Fourier transform has the property

$$V(e^{j\omega T}) = 0, \qquad \Omega_c < |\omega| \leqslant \pi. \tag{A.1}$$

Then the sequence $w(n)$ defined by

$$w(n) = v(n) - \sum_{k=1}^{N} a_k v(n-k) \tag{A.2}$$

also has a bandlimited Fourier transform. By setting $w(n) = 0$ for $1 \leqslant n \leqslant N$, we obtain the equations

$$v(n) = \sum_{k=1}^{N} a_k v(n-k), \qquad 1 \leqslant n \leqslant N. \tag{A.3}$$

These equations can be solved for the set of coefficients a_k, $k = 1, 2, \cdots, N$ such that $w(n) = 0$ for $1 \leqslant n \leqslant N$. In the special case where $v(n)$ is an autocorrelation function with bandlimited Fourier transform; i.e. $v(n) = v(-n)$ and

$$V(e^{j\omega T}) \geqslant 0 \tag{A.4}$$

then (A.3) are identical to the normal equations that arise in autoregressive spectrum analysis, and therefore they can be solved by the Levinson or Durbin recursion.

Thus we have shown that it is possible to construct a bandlimited sequence which is zero over an interval of N consecutive samples. This interval could obviously be positioned at any desired location by a shift.

Acknowledgment

We wish to acknowledge the contributions to this paper of R. Marucci through many discussions and for permission to use his example in Section IV-E. We also wish to thank Dr. R. W. Fink and Dr. W. Western for providing the gamma-ray spectrum data.

References

[1] P. Linz, *Theoretical Numerical Analysis*. New York: Wiley, 1979.
[2] A. W. Naylor and G. R. Sell, *Linear Operator Theory in Science and Engineering*. New York: Holt, Rinehart, and Winston, 1971.
[3] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*. New York: Academic Press, 1970.
[4] H. Landau, "On the recovery of a band-limited signal, after instantaneous companding and subsequent band limiting," *Bell Syst. Tech. J.*, vol. 39, pp. 351-364, Mar. 1960.
[5] H. J. Landau and W. L. Miranker, "The recovery of distorted band-limited signals," *J. Math Anal. Appl.*, vol. 2, pp. 97-104, 1961.
[6] G. Zames, "Nonlinear operators for system analysis," Tech. Rep. 370, Res. Lab. Electronics, M.I.T., Cambridge, MA, 1960.
[7] I. W. Sandberg, "On the properties of some systems that distort signals—Part I and II," Part I, *Bell Syst. Tech. J.*, vol. 42, pp. 2033-2047, Sept. 1963; Part II, *Bell Syst. Tech. J.*, vol. 43, pp. 91-112, Jan. 1964.
[8] W. K. Pratt, *Digital Image Processing*. New York: Wiley, 1978.
[9] A. Papoulis, "A new algorithm in spectral analysis and band-limited extrapolation," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 735-742, Sept. 1975.
[10] M. S. Sabri and W. Steenaart, "An approach to band-limited signal extrapolation: The extrapolation matrix," *IEEE Trans. Circuits Syst.*, vol. CAS-25, pp. 74-78, Feb. 1978.
[11] R. G. Wiley, "On an iterative technique for recovery of band-limited signals," *Proc. IEEE*, vol. 66, pp. 522-523, Apr. 1978.
[12] J. A. Cadzow, "An extrapolation procedure for band-limited signals," *IEEE Trans. Acoust., Speech, and Signal Processing*, vol. ASSP-27, pp. 4-12, Feb. 1979.
[13] L. A. Karlovitz, "Geometric methods in the existence and construction of fixed points of nonexpansive mappings," in *Constructive Approaches to Mathematical Models*, C. Coffman and G. Fix, Eds. New York: Academic Press, 1979, pp. 413-421.
[14] P. H. Van Cittert, "Zum Einfluss der Spaltbreite auf die Intensitatswerteilung in Spektrallinien II," *Z. für Physik*, vol. 69, pp. 298-308, 1931.
[15] B. R. Frieden, "Image enhancement and restoration," in *Picture Processing and Digital Filtering*, T. S. Huang, Ed. New York: Springer-Verlag, 1975, ch. 5, pp. 177-247.
[16] H. F. Silverman and A. E. Pearson, "On deconvolution using the discrete Fourier transform," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, pp. 112-118, Apr. 1973.
[17] R. W. Gerchberg, "Super-resolution through error energy reduction," *Optica Acta*, vol. 21, no. 9, pp. 709-720, 1974.
[18] R. Prost and R. Goutte, "Deconvolution when the convolution kernel has no inverse," *IEEE Trans. Acoust., Speech, and Signal Processing*, vol. ASSP-25, pp. 542-549, Dec. 1977.
[19] R. M. Mersereau and R. W. Schafer, "Comparative study of iterative deconvolution algorithms," in *Proc. 1978 IEEE Int. Conf. Acoustics, Speech and Signal Processing*, pp. 192-195, Apr. 1978.
[20] —, "Some techniques for digital deconvolution of positive constrained multi-dimensional sequences," in *Proc. 1978 European Conf. Circuit Theory and Design*, pp. 404-411, Sept. 1978.
[21] M. A. Richards, R. W. Schafer, and R. M. Mersereau, "An experimental study of the effects of noise on a class of iterative deconvolution algorithms," in *Proc. 1979 IEEE Int. Conf. Acoustics, Speech and Signal Processing*, pp. 401-404, Apr. 1979.
[22] R. W. Gerchberg and W. O. Saxton, "A practical algorithm for the determination of phase from image and diffraction plane pictures," *Optik*, vol. 35, no. 2, pp. 237-246, 1972.
[23] J. R. Fienup, "Reconstruction of an object from the modulus of its Fourier transform," *Optics Lett.*, vol. 3, no. 1, pp. 27-29, July 1978.
[24] J. R. Fienup, "Iterative method applied to image reconstruction and to computer-generated holograms," in *Proc. Soc. Photo-Optical Inst. Engineers*, 207-02, *Applications of Digital Image Processing III*, Aug. 27, 1979.
[25] M. H. Hayes, J. S. Lim and A. V. Oppenheim, "Phase-only signal restoration," in *Proc. 1980 Int. Conf. Acoustics, Speech, and Signal Processing*, pp. 437-440, Apr. 1980.
[26] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975.
[27] R. W. Schafer and L. R. Rabiner, "A digital signal processing approach to interpolation," *Proc. IEEE*, vol. 61, pp. 692-702, June 1973.
[28] P. A. Jansson, R. H. Hunt, and E. K. Pyler, "Resolution enhancement of spectra," *J. Opt. Soc. Amer.*, vol. 60, no. 5, pp. 596-599, May 1970.
[29] R. Marucci, M. S. Thesis, School of Electrical Engineering, Georgia Inst. Technol. (in preparation).
[30] G. T. Herman and A. Lent, "Iterative reconstruction algorithms," *Comput. Biol. Med.*, vol. 6, pp. 273-294, 1976.

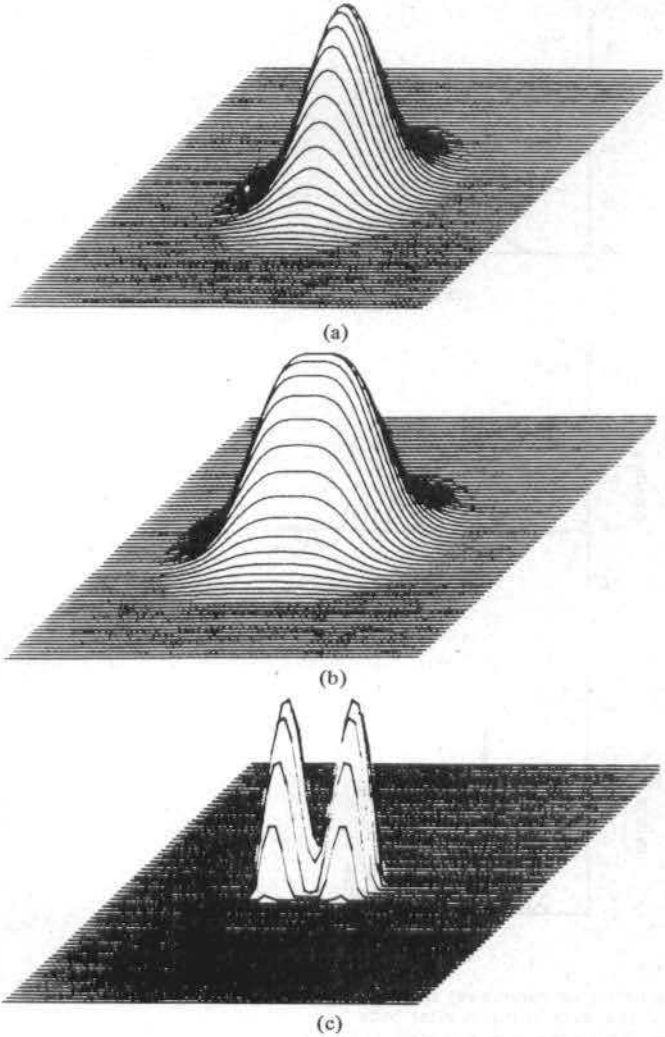

Fig. 5. Two-dimensional positive-constrained deconvolution of synthetic data. (a) Gaussian blurring function $h(m, n)$. (b) The sequence $y(m, n)$ obtained by convolving $h(m, n)$ with an impulse pair. (c) Estimate of the impulse pair obtained after 65 iterations with $\lambda = 2$.