AN EXPERIMENTAL STUDY OF THE EFFECTS OF NOISE ON A CLASS OF ITERATIVE DECONVOLUTION ALGORITHMS*

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ABSTRACT

There has recently been considerable interest in iterative deconvolution algorithms. Some of these algorithms seek to improve performance by incorporating á priori knowledge of the components of a convolution. For example, incorporation of both positivity and finite interval constraints tends to automatically introduce an extrapolation of the bandwidth. A question which naturally arises concerns the effect of noise on the performance of such algorithms. In this paper we describe an experimental evaluation of the effect of noise using synthetic data. Results of extensive tests are presented and some suggestions for mitigating the effects of noise are offered.

INTRODUCTION

Deconvolution is the process of recovering the input to a linear shift-invariant system given the output and knowledge of the system. This problem has arisen in a wide variety of applications and has long been a subject of research. The most successful algorithms for its solution incorporate á priori knowledge of the desired signal into the deconvolution process so as to improve performance. A serious difficulty with most techniques is their high sensitivity to noise in the data. In this paper we will describe the observed effects of noise on a class of iterative deconvolution algorithms and suggest a simple scheme for combatting these effects. Results will be illustrated with examples based on artificially constructed data.

A Class of Iterative Deconvolution Algorithms

The algorithms will be formulated for discrete one-dimensional signals, but are easily extended to higher-dimensional or continuous signals. Accordingly, assume that the given signal $y(n)$ is the sampled representation of a one-dimensional continuous signal $y_a(t)$, and that $h(n)$, the sampled representation of a bandlimited impulse response $h_a(t)$, is known. Assume also that $y(n)$ may be represented as

$$y(n) = x(n)*h(n) \tag{1}$$

where $x(n)$ is a bandlimited sampled representation of the desired signal $x_a(t)$ and the symbol * denotes discrete convolution. The deconvolution algorithms are described by the following equations:

$$x_o(n) = y(n) \tag{2a}$$

$$x_i(n) = \lambda y(n) + T[x_{i-1}(n)]*g(n) \quad , \; i=1,2,\ldots \tag{2b}$$

In these equations $x_i(n)$ is the ith approximation to $x(n)$, $g(n)$ is defined by

$$g(n) = \delta(n) - \lambda h(n) \tag{3}$$

where $\delta(n)$ is the unit sample sequence, λ is a parameter which controls the rate of convergence, and the operator T denotes an arbitrary transformation on the sequence $x_{i-1}(n)$. Algorithms included in this class differ mainly in the definition of the transformation T, and also in the choice of the parameter λ. This formulation includes a number of previously reported algorithms as special cases [4-7]. For example, Van Cittert's algorithm [4] results if $\lambda=1$ and $T[x_i]=x_i$. The choice of the operator T allows the easy incorporation of many types of á priori constraints into the deconvolution process.

A useful choice for T is the following:

$$T[x_{i-1}(n)] = x_i(n) \quad , \; n_1 \le n \le n_2 \, , \; x_i(n) \ge 0$$
$$= 0 \quad , \; \text{otherwise} \tag{4}$$

In this case the algorithm is tailored to the assumption that the desired signal is non-negative and of finite extent. This situation arises in several applications, such as the deconvolution of blurred line spectra or of light images. It has been shown that the algorithm converges in this case, subject to a proper choice of the parameter λ.

The discrete implementation of this procedure is straightforward, with one exception. Since the operator T is nonlinear, it tends to extend the bandwidth of the sequence it operates on. This tendency is reinforced by the filter $g(n)$, which is generally highpass in nature since $h(n)$ is lowpass in most cases. The net effect is to increase the bandwidth of $x_i(n)$ at each iteration. This bandwidth extrapolation phenomenon may imply a requirement for interpolation of the data $y(n)$

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and h(n), which can be accomplished using digital filtering techniques [8].

The performance of the algorithm in the noiseless case has been previously discussed [2,3]. It produces excellent results for data which can be modeled as the convolution of a group of positive impulses with a positive blurring function, such as blurred line spectra. An example is shown in Figure 1. Figure 1A is the convolution of two impulses of equal height and a separation of 8 samples with a Gaussian function having a standard deviation of 4 samples. This and all subsequent sequences have been interpolated by a factor of 4 using a linear phase FIR filter. Figure 1B shows the result obtained after 25 iterations of the algorithm with $\lambda=2$ and T chosen as in Eq. (4). The two peaks are clearly resolved, and the result is free of artifacts.

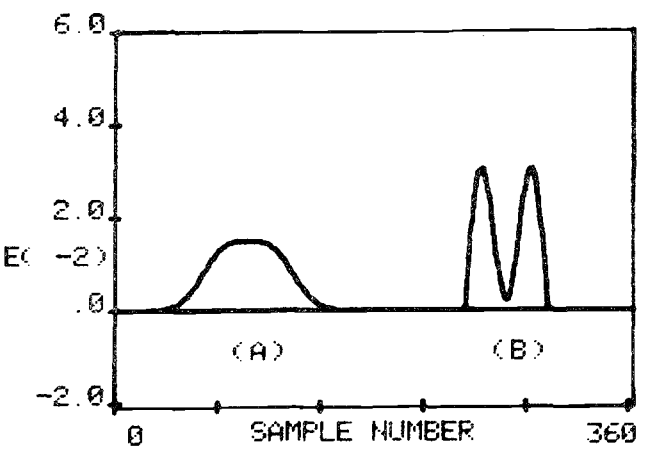

Figure 1. Deconvolution of Noiseless Data (A) Blurred Data (B) Result After 25 Iterations

THE EFFECTS OF NOISE

Assume that the sequence y(n) is corrupted by additive broadband random noise. Assume also that a relatively noiseless estimate of h(n) is available. Such an estimate is often available through auxiliary measurements. Furthermore, in many applications h(n) is approximately Gaussian in shape, and a Gaussian substitute may be used with little degradation of the results.

The addition of even moderate amounts of noise to the data has serious deleterious effects on the result. For blurred impulsive data, errors in the estimate of x(n) include splitting of single peaks into multiple peaks, the development of spurious peaks, and the obscuration of low-level peaks. These effects are illustrated in Figure 2. Figure 2A shows the data from Figure 1A corrupted by uniform pseudo-random noise with a signal-to-noise rato (SNR) of 30 db (SNR is defined as the ratio of the square of the maximum of the noiseless signal to the noise variance), while Figure 2B shows the result after deconvolution for 25 iterations. Peak splitting and spurious peaks are much in evidence.

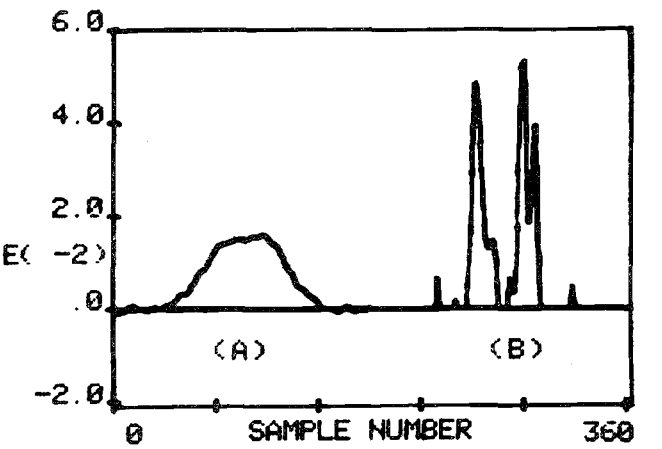

Figure 2. Deconvolution of Noisy Data (A) Blurred Data with 30 dB Noise (B) Result After 25 Iterations

A MODIFICATION TO IMPROVE PERFORMANCE ON NOISY DATA

Due to the lowpass nature of h(n) and thus of y(n), the SNR of noisy data is generally higher for low frequencies than for high frequencies. Since the deconvolution procedure tends to seek a solution by extrapolating from known spectral segments of x(n) to unknown segments, it might be possible to improve the result in the noisy case if the iteration was based only on the data from the lower frequency regions where the signal power, and thus SNR, is greatest. This observation leads to a simple method for combatting the effects of noise on the algorithm.

The modification to the algorithm consists of lowpass filtering both the data y(n) and the blurring function h(n) prior to applying the iterative procedure. The effect is to solve a substitute problem described by the relation

$$\begin{aligned} y'(n) &= h_f(n)*y(n) \\ &= h'(n)*x(n) \end{aligned} \tag{5}$$

where the new effective blurring function h'(n) is given by

$$h'(n) = h_f(n)*h(n) \tag{6}$$

and $h_f(n)$ is the unit-sample response of the lowpass filter applied to y(n) and h(n). The sequences y'(n) and h'(n) are now used in place of y(n) and h(n) in the procedure. However, whereas y(n) contains broadband noise, y'(n) contains relatively narrowband noise of less total power,

much of the noise having been removed by the filtering process. This idea is easily extended to deal with bandpass noise or more general blurring functions. Its efficacy is a function of the concentration in frequency of the noiseless signal y(n). An additional benefit of this procedure derives from the substitution of h'(n) for h(n). If the estimate of h(n) is corrupted by broadband noise, then the noise will be reduced by the filtering process.

The implementation of this modification adds very little to the computation involved in deconvolution. The filtering of y(n) and h(n) is required only once prior to initiating the iteration. If interpolation of the data is also required, as is often the case, the two filtering operations may be combined into one.

The effect of this modification to the algorithm is illustrated in Figure 3, which continues the example of Figure 2. This shows the result obtained after 25 iterations when the noisy data is prefiltered with a lowpass filter. The cutoff frequency is expressed as a fraction of the total possible bandwidth of the data (BWF). As the passband is made narrower, the degree of peak splitting and the number and intensity of spurious peaks are reduced, so that the result more closely approaches the noiseless case.

Effect on Convergence

As the passband of the lowpass filter is made narrower, a greater degree of noise rejection is obtained. At the same time, however, the algorithm is provided with information about x(n) over a more limited portion of the spectrum so that a greater degree of bandwidth extrapolation is required. When the filtering becomes so severe as to remove spectral regions in which y(n) has significant energy, increasing numbers of iterations are required to maintain a given resolution in the result. Thus, the improved performance on noisy data is obtained at the cost of greater computation.

This phenomenon is evident in Figure 3D, where 90% of the spectrum of the original noisy data was removed prior to deconvolution. The resolution of the result is poor compared to the other cases. The tradeoff between noise rejection and resolution is summarized in Figure 4. This graph was obtained using data resulting from the deconvolution of a single blurred impulse in the absence of noise. It depicts the normalized width at half the maximum value of the deconvolved peak as a function of the number of iterations, and clearly shows that restricting the passband of the noise rejection filter slows the rate of convergence.

Also shown in Figure 4 is the effect of the parameter λ upon the rate of convergence. For $\lambda=2$, the number of iterations required to achieve a given resolution is about one half that required when $\lambda=1$. The choice of $\lambda=2$ appears to be nearly optimal when h(n) is normalized so that $|H(e^{j\omega})|$ has a maximum value of one. Choosing λ smaller slows convergence, while choosing it larger leads to instability of the algorithm.

CONCLUSIONS

We have described a simple filtering technique for improving the performance of a class of iterative deconvolution algorithms when the data is subject to broadband noise. The effectiveness of the procedure was demonstrated, and the computational tradeoffs involved were discussed. Although the results were illustrated for only one member of the class, similar results might be expected to hold for other members which exhibit the bandwidth extrapolation property. Present research concerns the effects of other deviations from the signal model (for example, a poor estimate of h(n)) and the comparative performance of this procedure to other positive constrained algorithms.

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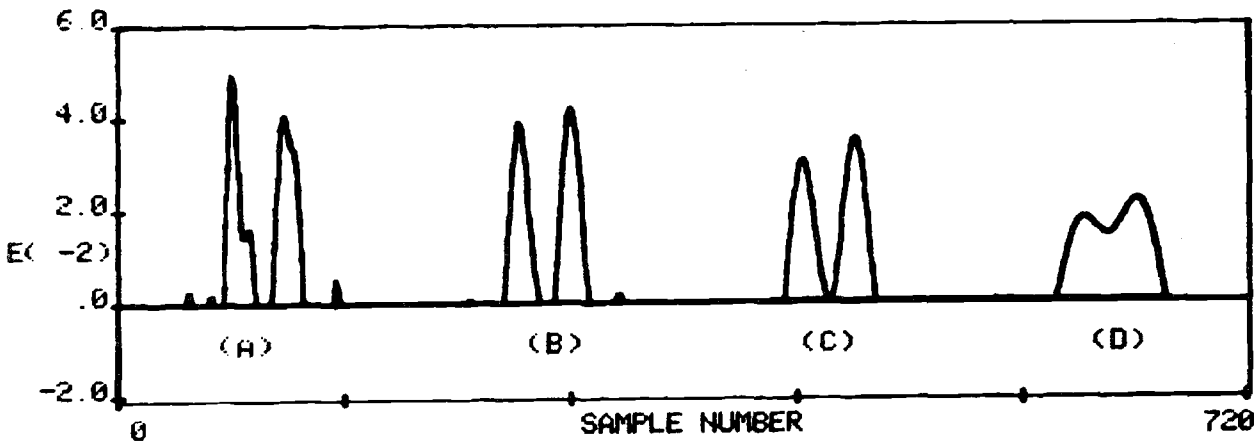

Figure 3. Deconvolution of Noisy Data Using Varying Degrees of Prefiltering. The Result After 25 Iterations is Shown. (A) BWF=0.75 (B) BWF=0.5 (C) BWF=0.25 (D) BWF=0.1.

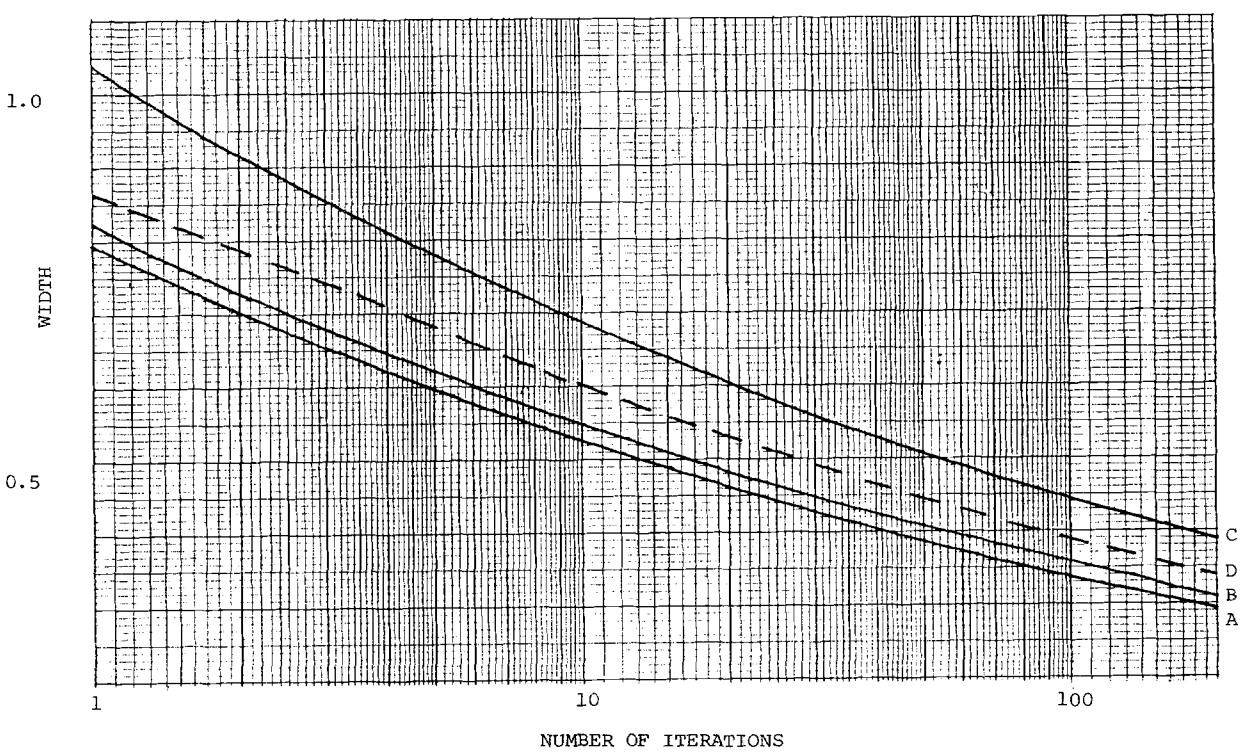

Figure 4. Effect of Prefiltering and λ on Rate of Convergence. (A) λ=2, BWF=1.0,0.5, (B) λ=2, BWF=0.25, (C) λ=2, BWF=0.1, (D) λ=1, BWF=1.0.